M2 - ВЕЖБЕ - 8. термин

Драган Ђорић

Факултет организационих наука

2009/2010

Неодређени интеграл (НИ)

$F:(a,b)\to R$ је *примитивна функција* за f ако је

$$\forall x\in(a,b)\ \ F'(x)=f(x)$$

$$\int f(x)dx=\{F \mid F \text{ je prim. funkcija za } f\}=\{F+C,\ C\in R\}=F(x)+C$$

F	$\sin x$	e^x	$-e^{-x}$	$\arctan x$	$\ln x$	$\frac{1}{n+1}x^{n+1}$	$\cdots$
f	$\cos x$	e^x	e^{-x}	$\frac{1}{1+x^2}$	$\frac{1}{x}$	x^n	$\cdots$

Свака непрекидна функција има примитивну

Примитивна функција елементарне функције не мора бити елементарна функција. На пример, за функције

$$x\mapsto\frac{\sin x}{x},\qquad x\mapsto\frac{e^x}{x},\qquad x\mapsto e^{-x^2},\qquad x\mapsto\sin x^2$$

примитивне функције нису елементарне

1 Збирка, зад.10, стр.73

За функцију $f : x \mapsto \frac{1}{1+x^2}$ одредити примитивну F за коју је $F(1) = \pi$.

Решење.

Једна примитивна је $\arctan x$

Скуп свих примитивних је $\{\arctan x + C,\ C \in R\}$

Из услова $\arctan 1 + C = \pi$ следи $C = \frac{3\pi}{4}$

$F(x) = \arctan x + \frac{3\pi}{4}$

2 Збирка, зад.4, стр.73

Одредити бар једну примитивну функцију за функцију $f : x \mapsto e^{|x|}$

Решење.

$$e^{|x|} = \begin{cases} e^{x}, & x \geq 0 \\ e^{-x}, & x < 0 \end{cases}$$

За $x > 0$ примитивне су $e^{x} + A,\ A \in R$

За $x < 0$ примитивне су $-e^{-x} + B,\ B \in R$

За $x \in R$ примитивна функција F мора бити диференцијабилна (дакле, и непрекидна) у тачки $x = 0$

На пример,

$$F(x) = \begin{cases} e^{x} - 1, & x \geq 0 \\ -e^{-x} + 1, & x < 0 \end{cases}$$

3

Доказати да су $F : x \mapsto \ln(x + \sqrt{x^2 + x^2})$ $(a \neq 0)$ и $G : x \mapsto \ln(x + \sqrt{x^2 - a^2})$ $(x \geq a > 0)$ примитивне функције за функције $f : x \mapsto \dfrac{1}{\sqrt{x^2 + a^2}}$ и $g : x \mapsto \dfrac{1}{\sqrt{x^2 - a^2}}$

Решење.

$$F'(x) = \frac{1 + \frac{2x}{2\sqrt{x^2+a^2}}}{x + \sqrt{x^2 + a^2}} = \frac{\sqrt{x^2 + a^2} + x}{\sqrt{x^2 + a^2}} \cdot \frac{1}{x + \sqrt{x^2 + a^2}} = \frac{1}{\sqrt{x^2 + a^2}} = f(x)$$

Слично је и $G'(x) = g(x)$

Уврстити у табличне $\displaystyle\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln(x + \sqrt{x^2 \pm a^2})$

Својства интеграла

$$\int k \cdot f(x)dx = k \cdot \int f(x)dx \;\; (\text{хомогеност})$$

$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx \;\; (\text{адитивност})$$

$$\int \sum_{i=1}^{n} c_i f_i(x)dx = \sum_{i=1}^{n} c_i \int f_i(x)dx \;\; (\text{линеарност})$$

Налажење интеграла – Таблица + својства

4 Збирка, зад.14, стр.74

$f(x) = \dfrac{(x-1)(x^2+2)}{5x^2}, \; I = \displaystyle\int f(x)dx = ?$

Решење.

$$f(x) = \frac{1}{5x^2}(x^3 - x^2 + 2x - 2) = \frac{1}{5}x - \frac{1}{5} + \frac{2}{5}\cdot\frac{1}{x} - \frac{2}{5}\cdot\frac{1}{x^2}$$

За $\alpha \neq -1$ je $\displaystyle\int x^\alpha dx = \frac{1}{\alpha+1}x^{\alpha+1} + C, \; \int \frac{dx}{x} = \ln|x| + C$

$$\begin{aligned} I &= \frac{1}{5}\cdot\frac{1}{2}x^2 - \frac{1}{5}x + \frac{2}{5}\ln|x| - \frac{2}{5}\cdot\frac{-1}{x} + C \\ &= \frac{1}{10}x^2 - \frac{1}{5}x + \frac{2}{5}\ln|x| + \frac{2}{5}\cdot\frac{1}{x} + C \end{aligned}$$

5 Збирка, зад.15, стр.74

$f(x) = \dfrac{x^2}{1+x^2}$, $I = \int f(x)dx = ?$

Решење.

$$f(x) = \frac{x^2+1-1}{1+x^2} = \frac{x^2+1}{1+x^2} - \frac{1}{1+x^2} = 1 - \frac{1}{1+x^2}$$

$\displaystyle\int \frac{dx}{1+x^2} = \arctan x + C$

$$I = x - \arctan x + C$$

6 Збирка, зад.16, стр.74

$f(x) = \tan^2 x,\ I = \int f(x)dx = ?$

Решење.

$$f(x) = \frac{\sin^2 x}{\cos^2 x} = \frac{1 - \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} - 1$$

$\int \frac{dx}{\cos^2 x} = \tan x + C$

$$I = \tan x - x + C$$

7 Збирка, зад.18, стр.74

$f(x) = \dfrac{1}{\sin^2 x \cos^2 x}, \; I = \displaystyle\int f(x)dx = ?$

Решење.

$$f(x) = \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} = \frac{1}{\cos^2 x} + \frac{1}{\sin^2} x$$

$\displaystyle\int \frac{dx}{\sin^2 x} = -\cot x + C$

$$I = \tan x - \cot x + C$$

8

$$f(x)=\frac{1}{a^2-x^2},\ I=\int f(x)dx=?$$

Решење.

$$\begin{aligned}
f(x) &= \frac{1}{(a-x)(a+x)}=\frac{1}{2a}\cdot\frac{a+x+a-x}{(a-x)(a+x)}\\
&= \frac{1}{2a}\cdot\frac{1}{a-x}+\frac{1}{2a}\cdot\frac{dx}{a+x}\\
&= \frac{1}{2a}\cdot\frac{1}{a+x}-\frac{1}{2a}\cdot\frac{dx}{x-a}
\end{aligned}$$

$$I=\frac{1}{2a}\int\frac{dx}{a+x}-\frac{1}{2a}\int\frac{dx}{x-a}=\frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right|+C$$

Уврстити у табличне $\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right|+C$

9

$$f(x) = \frac{1}{(1-x^2)(1+x^2)}, \; I = \int f(x)dx = ?$$

Решење.

$$f(x) = \frac{1}{2} \cdot \frac{1+x^2+1-x^2}{(1-x^2)(1+x^2)} = \frac{1}{2} \cdot \frac{1}{1-x^2} + \frac{1}{2} \cdot \frac{1}{1+x^2}$$

$$I = \frac{1}{2}\int \frac{dx}{1-x^2} + \frac{1}{2}\int \frac{dx}{1+x^2} = \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| + \frac{1}{2}\arctan x + C$$

Замена променљиве

Ако је

$$\int f(x)dx = F(x) + C,$$

тада је

$$\int f(u(x))du(x) = F(u(x)) + C$$

На пример,

из $\int \frac{dx}{x} = \ln|x| + C$ имамо $\int \frac{du(x)}{u(x)} = \ln|u(x)| + C$

из $\int \frac{dx}{1+x^2} = \arctan x + C$ имамо $\int \frac{du(x)}{1+u^2(x)} = \arctan u(x) + C$

$$f(x) = \frac{1}{a^2 + x^2} \ \ (a \neq 0), \ \ I = \int f(x)dx = ?$$

Решење.

$$I = \frac{1}{a} \int \frac{d(x/a)}{1 + (x/a)^2} = \frac{1}{a} \int \frac{du}{1 + u^2} = \frac{1}{a} \arctan u + C$$

Дакле, $I = \frac{1}{a} \arctan \frac{x}{a} + C$

Уврстити у табличне $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a} + C$

$$f(x) = \frac{1}{\sqrt{a^2 - x^2}} \quad (a > 0), \quad I = \int f(x)dx = ?$$

Решење.

$$I = \frac{1}{a}\int \frac{d(x)}{\sqrt{1-(x/a)^2}} = \int \frac{d(x/a)}{\sqrt{1-(x/a)^2}} = \int \frac{du}{\sqrt{1-u^2}} = \arcsin u + C$$

Дакле, $I = \arcsin\frac{x}{a} + C$

Уврстити у табличне $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C$

12 Збирка, зад.18, стр.74

$f(x) = \tan x,\; I = \int f(x)dx =?$

Решење.

$$I = \int \frac{\sin x}{\cos x dx} = -\int \frac{d(\cos x)}{\cos x} = -\int \frac{du(x)}{u(x)}, \qquad u(x) = \cos x$$

$$I = -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos x| + C$$

13 Збирка, зад.35, стр.74

$f(x) = \dfrac{\sin 2x}{1+\sin^2 x}$, $I = \int f(x)dx = ?$

Решење.

$$\int f(x)dx = \int \frac{2\sin x\cos x}{1+\sin^2 x}dx = \int \frac{d(1+\sin^2 x)}{1+\sin^2 x} = \int \frac{du}{u}$$

$$I = \int \frac{du}{u} = \ln|u| + C = \ln(1+\sin^2 x) + C$$

14

$$I = \int x e^{-x^2} dx = ?$$

Решење.

$$I = -\frac{1}{2} \int e^{-x^2} d(-x^2) = -\frac{1}{2} \int e^u du = -\frac{1}{2} e^u + C = -\frac{1}{2} e^{-x^2} + C$$

15

$I = \int \frac{1}{x^2} \sin \frac{1}{x} = ?$

Решење.

$$I = -\int \sin \frac{1}{x} d\left(\frac{1}{x}\right) = -\int \sin u du = \cos u + C = \cos \frac{1}{x} + C$$

16 Збирка, зад.38, стр.74

$$f(x) = \sin^3 x \cos x dx, \; I = \int f(x)dx = ?$$

Решење.

$$\int f(x)dx = \int \sin^3 x d(\sin x) = \int u^3 du$$

$$I = \int u^3 du = \frac{1}{4}u^4 + C = \frac{\sin^4 x}{4} + C$$

17 Збирка, зад.38, стр.74

$$I = \int \frac{\arctan^2 x}{1+x^2} dx = ?$$

Решење.

$$I = \int \arctan^2 x d(\arctan x) = \int u^2 du = \frac{1}{3}u^3 + C = \frac{1}{3}\arctan^3 x + C$$

18 Збирка, зад.43, стр.75

$$I = \int \frac{x dx}{1 + x^4} = ?$$

Решење.

$$I = \frac{1}{2} \int \frac{d(x^2)}{1 + (x^2)^2} = \frac{1}{2} \int \frac{du}{1 + u^2} = \frac{1}{2} \arctan u + C = \frac{1}{2} \arctan x^2 + C$$

19 Збирка, зад.46, стр.75

$$I = \int \frac{x^3 dx}{\sqrt{1-x^8}} = ?$$

Решење.

$$I = \frac{1}{4}\int \frac{d(x^4)}{\sqrt{1-(x^4)^2}} = \frac{1}{4}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{4}\arcsin u + C = \frac{1}{4}\arcsin x^4 + C$$

20 Збирка, зад.47, стр.75

$$I = \int \frac{dx}{\sqrt{x(1-x)}} = ?$$

Решење.

$$I = \int \frac{dx}{\sqrt{x}\sqrt{1-x}} = 2\int \frac{d(\sqrt{x})}{\sqrt{1-(\sqrt{x})^2}} == 2\int \frac{du}{\sqrt{1-u^2}}$$

$$I = 2\arcsin u + C = 2\arcsin\sqrt{x} + C$$

21 Збирка, зад.45, стр.75

$$I = \int \frac{dx}{a^2\cos^2 x + b^2 \sin^2 x} = ?$$

Решење.

$$I = \int \frac{dx}{\cos^2 x(a^2 + b^2\tan^2 x)} = \frac{1}{a^2}\int \frac{d(\tan x)}{1 + \left(\frac{b}{a}\tan x\right)^2} = \frac{1}{a^2}\cdot\frac{a}{b}\int \frac{d\left(\frac{b}{a}\tan x\right)}{1 + \left(\frac{b}{a}\tan x\right)^2}$$

$$I = \frac{1}{ab}\int \frac{du}{1+u^2} = \frac{1}{ab}\arctan u + C = \frac{1}{ab}\arctan\left(\frac{b\tan x}{a}\right) + C$$

Метода смене

$x = \varphi(t)$, φ - диференцијабилна

$$\int f(x)dx = \int f(\varphi(t))\varphi'(t)dt = \int g(t)dt = G(t) + C = G(\varphi^{-1}(x)) + C$$

$F(x) = G(\varphi^{-1}(x))$ ако постоји φ^{-1}

22 Збирка, зад.61, стр.75

$$I = \int \frac{dx}{\sqrt{x}(1 + \sqrt[3]{x})}$$

Решење.

$$x = t^6 = \varphi(t), \;\; \sqrt{x} = t^3, \;\; \sqrt[3]{x} = t^2, \;\; dx = 6t^5 dt, \;\; \varphi^{-1} : x \mapsto \sqrt[6]{x}$$

$$I = \int \frac{6t^5 dt}{t^3(1 + t^2)} = 6 \int \frac{t^2 dt}{1 + t^2} = 6 \int \frac{1 + t^2}{1 + t^2} dt - 6 \int \frac{dt}{1 + t^2}$$

$$I = 6t - 6 \arctan t + C = 6(\sqrt[6]{x} - \arctan \sqrt[6]{x}) + C$$

23

$$I = \int \frac{dx}{(1+x)\sqrt{x}}$$

Решење.

$x = t^2,\ \ dx = 2tdt,\ \ t = \sqrt{x}$

$$I = \int \frac{2tdt}{(1+t^2)t} = 2\int \frac{dt}{1+t^2} = 2\arctan t + A = 2\arctan\sqrt{x} + A = F(x) + A$$

$x = 1/t^2,\ \ dx = -2dt/t^3,\ \ t = 1/\sqrt{x}$ (друга смена)

$$I = -2\frac{dt}{1+t^2} = -2\arctan t + B = -2\arctan\frac{1}{\sqrt{x}} + B = G(x) + B$$

$F(x) = G(x) + C$, односно $2\arctan\sqrt{x} = -2\arctan\dfrac{1}{\sqrt{x}} + C$

За $x = 1:\ \ 2\cdot\dfrac{\pi}{4} = -2\cdot\dfrac{\pi}{4} + C,\ \ C = \pi$ Дакле, важи једнакост

$$\arctan\sqrt{x} + \arctan\frac{1}{\sqrt{x}} = \frac{\pi}{2}$$

24 Збирка, зад.64, стр.76

$$I = \int \frac{dx}{x^2\sqrt{1+x^2}}$$

Решење.

$$x = 1/t,\;\; dx = -dt/t^2,\;\; 1+x^2 = 1+1/t^2 = \frac{t^2+1}{t^2}$$

$$I = -\int \frac{dt}{t^2}\cdot\frac{t^2\cdot t}{\sqrt{1+t^2}} = -\int \frac{tdt}{\sqrt{1+t^2}}$$

$$I = -\frac{1}{2}\int \frac{d(1+t^2)}{\sqrt{1+t^2}} = -\sqrt{1+t^2}+C = -\sqrt{1+1/x^2}+C$$

25 Збирка, зад.76, стр.76

$$I = \int \sqrt{a^2 - x^2}dx, \;\; a > 0$$

Решење.

$x = a\sin t = \varphi(t), \;\; \varphi$ - на $(-\pi/2, \pi/2)$ има инверзну (монотона је)

$$\varphi^{-1} : x \mapsto t = \arcsin\frac{x}{a}$$

$$I = \int \sqrt{a^2 - a^2\sin^2 t} \cdot a\cos t dt = a^2 \int \cos^2 t dt = \frac{a^2}{2}\int (1 + \cos 2t)dt$$

$$I = \frac{a^2}{2}\left(t + \frac{1}{2}\sin 2t\right) + C = \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + C$$

Смена $\psi(x) = t$

Ако је $f(x) = g(\psi(x))$, тада сменом $\psi(x) = t$ имамо

$$\int f(x)dx = \int g(\psi(x))dx = \int g(t)d(\psi^{-1}(t)) = \int h(t)dt = H(t) + C$$

Дакле,

$$\int f(x)dx = H(\psi(x)) + C$$

26

$$I = \int e^{e^{-x}-x} dx$$

Решење.

$$\psi(x) = e^{-x} = t, \ \ d(e^{-x}) = dt, \ \ -e^{-x} dx = dt$$

$$I = \int e^{e^{-x}} \cdot \underbrace{e^{-x} dx}_{-dt} = -\int e^t dt = -e^t + C$$

$$I = -e^{e^{-x}} + C$$

27 Збирка, зад.71, стр.76

$$I = \int \frac{\cos(\ln x)}{x} dx$$

Решење.

$$\psi(x) = \ln x = t, \;\; dx/x = dt$$

$$I = \int \cos(\ln x) \frac{dx}{x} = \int \cos t dt = \sin t + C = \sin(\ln x) + C$$

28

$$I = \int \frac{dx}{1+e^x}$$

Решење.

$$\psi(x) = 1 + e^x = t, \;\; e^x = t-1, \;\; x = \ln(t-1), \;\; dx = \frac{dt}{t-1}$$

$$I = \int \frac{dt}{t(t-1)} = \int \frac{t-(t-1)}{t(t-1)} dt = \int \frac{dt}{t-1} - \int \frac{dt}{t} = \ln|t-1| - \ln|t| + C$$

$$I = \ln \frac{e^x}{1+e^x} + C$$

29 Збирка, зад.74, стр.76

$$I = \int \frac{1}{1-x^2} \ln \frac{1+x}{1-x} dx$$

Решење.

$$\ln \frac{1+x}{1-x} = t, \;\; d\left(\ln \frac{1+x}{1-x}\right) = dt, \;\; \frac{1-x}{1+x} \cdot \frac{1-x+(1+x)}{(1-x)^2} dx = dt, \;\; \frac{2dx}{1-x^2} = dt$$

$$I = \frac{1}{2} \int t dt = \frac{1}{2} \cdot \frac{t^2}{2} = \frac{1}{4} \ln^2 \frac{1+x}{1-x} + C$$

30 Збирка, зад.75, стр.76

$$I = \int \frac{\arctan \sqrt{x}}{\sqrt{x}(1+x)} dx$$

Решење.

$$\arctan \sqrt{x} = t, \;\; d(\arctan \sqrt{x}) = dt, \;\; \frac{1}{1+x} \cdot \frac{dx}{2\sqrt{x}} = dt$$

$$I = \int t \cdot 2dt = t^2 + C = \arctan^2 \sqrt{x} + C$$

M2 - ВЕЖБЕ - 9. термин

Драган Ђорић

Факултет организационих наука

2009/2010

Садржај

1. Метода парцијалне интеграције

2. Интеграција рационалних функција

Метода парцијалне интеграција

$$d(uv) = udv + vdu$$

$$\int d(uv) = \int udv + \int vdu$$

$$\int udv = uv - \int vdu$$

Корсити се за $\int x^n \cdot \{\ln x, e^{ax}, \sin ax, \cos ax, \arcsin ax, \arctan ax\}\, dx$

$$\int e^{ax} \cdot \{\sin bx, \cos bx\}\, dx$$

1 Збирка, зад.90, стр.77

$I = \int \frac{x^2 dx}{(1+x^2)^2} = ?$

Решење.

$u = x$ и $dv = \frac{x dx}{(1+x^2)^2}$

$du = dx$ и $v = -\frac{1}{2} \cdot \frac{1}{1+x^2}$,

$$I = -\frac{x}{2(1+x^2)} + \frac{1}{2}\int \frac{dx}{1+x^2} = -\frac{x}{2(1+x^2)} + \frac{1}{2}\arctan x + C.$$

$$I_2 = \int \frac{dx}{(x^2+a^2)^2} = ?$$

Решење.

$$I_1 = \int \frac{dx}{x^2+a^2}$$

Ако за интеграл I_1 применимо парцијалну интеграцију $(dv = dx)$, добијамо

$$I_1 = \frac{x}{x^2+a^2} + 2\int \frac{x^2dx}{(x^2+a^2)^2} = \frac{x}{x^2+a^2} + 2I_1 - 2a^2 I_2$$

$$I_2 = \frac{1}{2a^2} \cdot \frac{x}{x^2+a^2} + \frac{1}{2a^2} I_1 = \frac{1}{2a^2} \cdot \frac{x}{x^2+a^2} + \frac{1}{2a^3} \operatorname{arctan} \frac{x}{a} + C$$

3

$$I = \int \sqrt{a^2 + x^2}dx = ?$$

Решење.

$$u = \sqrt{a^2 + x^2}, \;\; dv = dx$$

$$du = \frac{x}{\sqrt{a^2 + x^2}}, \;\; v = x$$

$$\begin{aligned} I &= x\sqrt{a^2 + x^2} - \int \frac{x^2 dx}{\sqrt{a^2 + x^2}} = x\sqrt{a^2 + x^2} - \int \frac{a^2 + x^2 - a^2}{\sqrt{a^2 + x^2}}dx \\ &= x\sqrt{a^2 + x^2} - I + a^2 \ln|x + \sqrt{a^2 + x^2}| + C \end{aligned}$$

$$I = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\ln|x + \sqrt{a^2 + x^2}| + C$$

$$I = \int \frac{x}{\sqrt{1-x^2}} \ln \frac{1+x}{1-x} dx = ?$$

Решење.

$$u = \ln \frac{1+x}{1-x}, \;\; dv = \frac{x dx}{\sqrt{1-x^2}}$$

$$du = \frac{1-x}{1+x} \cdot \frac{1-x+1+x}{(1-x)^2} dx, \qquad v = -\frac{1}{2} \int \frac{d(1-x^2)}{\sqrt{1-x^2}} = -\sqrt{1-x^2}$$

$$I = -\sqrt{1-x^2} \ln \frac{1+x}{1-x} + 2 \int \frac{dx}{\sqrt{1-x^2}} = -\sqrt{1-x^2} \ln \frac{1+x}{1-x} + 2 \arcsin x + C$$

5 Збирка, зад.113, стр.78

$$I = \int \arcsin^2 x dx = ?$$

Решење.

$$u = \arcsin^2 x,\ dv = dx,\ du = 2\frac{\arcsin x}{\sqrt{1-x^2}}dx,\ \ v = x$$

$$I = x \cdot \arcsin^2 x - 2J, \qquad J = \int \frac{x \arcsin x}{\sqrt{1-x^2}}dx$$

За J: $u = \arcsin x$ и $dv = \dfrac{xdx}{\sqrt{1-x^2}}$

$$du = \frac{dx}{\sqrt{1-x^2}} \text{ и } v = -\sqrt{1-x^2}$$

$$J = -\sqrt{1-x^2}\arcsin x + x + C$$

$$I = x \cdot \arcsin^2 x + 2\sqrt{1-x^2}\arcsin x - 2x + C$$

6 Збирка, зад.110, стр.78

$$I = \int x^2 \arccos x dx = ?$$

Решење.

$u = x^2$ **и** $dv = \arccos x dx$

$$v = \int \arccos x dx = x \arccos x + \int \frac{xdx}{\sqrt{1-x^2}} = x \arccos x - \sqrt{1-x^2}$$

$$I = x^3 \arccos x - x^2\sqrt{1-x^2} - 2I + 2\int x\sqrt{1-x^2}dx.$$

$$3I = x^3 \arccos x - x^2\sqrt{1-x^2} + 2\int x\sqrt{1-x^2}dx$$

$$3I = x^3 \arccos x - x^2\sqrt{1-x^2} - \frac{1}{3}\sqrt{(1-x^2)^3}$$

7 Колоквијум, 2002 (Збирка, Пр.9)

$I_n = \int \arcsin^n x dx$, Веза I_n и I_{n-2} ?

Решење.

$u = \arcsin^n x, \; dv = dx$

$$I_n = x \cdot \arcsin^n x - n \int \frac{x \cdot \arcsin^{n-1} x}{\sqrt{1-x^2}} dx = x \cdot \arcsin^n x - n \cdot J$$

За J : $\quad u = \arcsin^{n-1} x, \; dv = \dfrac{x}{\sqrt{1-x^2}}, \quad v = -\sqrt{1-x^2}$

$$J = -\sqrt{1-x^2} \cdot \arcsin^{n-1} x + (n-1) I_{n-2}$$

$$I_n = x \cdot \arcsin^n x + n\sqrt{1-x^2} \cdot \arcsin^{n-1} x - n(n-1) I_{n-2}$$

Интеграција рационалних функција

$$f(x) = \frac{P(x)}{Q(x)}, \qquad st(P) < st(Q)$$

Ако је

$$Q(x) = a_n(x - a_1)^{k_1}(x - a_2)^{k_2} \cdots (x - a_{k_l})^{k_l}$$

тада сваком фактору $(x - a)^k$ одговара збир парцијалних разломака

$$\frac{A_1}{x - a} + \frac{A_2}{(x - a)^2} + \cdots + \frac{A_k}{(x - a)^k}$$

$$\int \frac{xdx}{(x+1)(x+2)(x+3)}$$

Решење.

$$\frac{x}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$$

$$x = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$$

За $x = -1$ следи $A = -1/2$

За $x = -2$ следи $B = -2$

За $x = -3$ следи $C = -3/2$

$$
\begin{aligned}
I &= -\frac{1}{2}\int \frac{dx}{x+1} + 2\int \frac{dx}{x+2} - \frac{3}{2}\int \frac{dx}{x+3} \\
&= -\frac{1}{2}\ln|x+1| + 2\ln|x+2| - \frac{2}{3}\ln|x+3| + C \\
&= \ln\frac{(x+2)^2}{|x+1|^{1/2}|x+3|^{3/2}} + C
\end{aligned}
$$

9 Збирка, зад.132, стр.79

$$I = \int f(x)dx, \;\; f(x) = \frac{x^4 - 3x^2 - 3x - 2}{x^3 - x^2 - 2x}$$

Решење.

$$f(x) = \frac{x^4 - x^3 - 2x^2 + x^3 - x^2 - 2x - x - 2}{x^3 - x^2 - 2x} = x + 1 - \frac{x+2}{x(x^2 - x - 2)}$$

$$\frac{x+2}{x(x-2)(x+1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1}$$

$$x + 2 = A(x-2)(x+1) + Bx(x+1) + Cx(x-2)$$

За $x = 0, \;\; 2 = A \cdot (-2) \cdot 1, \;\; A = -1$

За $x = -1, \;\; 1 = C \cdot (-1) \cdot (-3), \;\; C = 1/3$

За $x = 2, \;\; 4 = B \cdot 2 \cdot 3, \;\; B = 2/3$

$$I = \frac{x^2}{2} + x + \int \frac{dx}{x} - \frac{2}{3} \int \frac{dx}{x-2} - \frac{1}{3} \int \frac{dx}{x+1}$$

$$I = \frac{x^2}{2} + x + \ln|x| - \frac{2}{3} \ln|x-2| - \frac{1}{3} \ln|x+1| + C$$

10

$$I=\int\frac{x^2+5x-2}{(x^2-1)(x+1)}dx=?$$

Решење.

$$\frac{x^2+5x-2}{(x^2-1)(x+1)}=\frac{x^2+5x-2}{(x-1)(x+1)^2}=\frac{A}{x-1}+\frac{B}{(x+1)^2}+\frac{C}{x+1}$$

За $x=1$ је $1^2+5\cdot1-2=A\cdot(1+1)^2,\ 4A=4,\ A=1$

За $x=-1$ је $(-1)^2+5\cdot(-1)-2=B\cdot(-1-1),\ B=3$

За $x=0$ је $-2=A-B-C,\ -2=1-3-C,\ C=0$

$$I=\int\frac{dx}{x-1}+3\cdot\int\frac{dx}{(x+1)^2}=\ln|x-1|-\frac{3}{x+1}+C$$

11 Збирка, зад.133, стр.79

$$I = \int \frac{5x-8}{x^3-4x^2+4x}dx = ?$$

Решење.

$$\frac{5x-8}{x^3-4x^2+4x} = \frac{5x-8}{x(x-2)^2} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$5x-8 = A(x-2)^2 + Bx(x-2) + Cx$$

За $x=0, \quad -8 = A\cdot 4, \ A=-2;$ За $x=2, \quad 2 = C\cdot 2, \ C=1$

$$5x-8 = -2(x-2)^2 + Bx(x-2) + x$$

За $x=1, \ -3 = -2 + B\cdot(-1) + 1, \ -2 = B\cdot(-1), \ B=2$

$$I = -2\int\frac{dx}{x} + 2\int\frac{dx}{x-2} + \int\frac{dx}{(x-2)^2} = -2\ln|x| + 2\ln|x-2| - \frac{1}{x-2} + C$$

$$I = \ln\left(\frac{x-2}{x}\right)^2 - \frac{1}{x-2} + C$$

12

$$I = \int \frac{dx}{x^3(x+1)^3} = ?$$

Решење.

$$\frac{1}{x^3(x+1)^3} = \frac{A}{x^3} + \frac{B}{x^2} + \frac{C}{x} + \frac{D}{(x+1)^3} + \frac{E}{(x+1)^2} + \frac{F}{x+1}$$

$$A(x+1)^3 + Bx(x+1)^2 + Cx^2(x+1)^3 + Dx^3 + Ex^3(x+1) + Fx^3(x+1)^2 = 1.$$

За $x = 0$ добијамо $A = 1$, а за $x = -1$ добијамо $D = -1$

$$Bx(x+1)^3 + Cx^2(x+1)^3 + Ex^3(x+1) + Fx^3(x+1)^2 = -3x(x+1)$$

$$B(x+1)^2 + Cx(x+1)^2 + Ex^2 + Fx^2(x+1) = -3$$

За $x = 0$ добијамо $B = -3$, а за $x = -1$ добијамо $E = -3$

$$Cx(x+1)^2 + Fx^2(x+1) = 6x(x+1)$$

$$C(x+1) + Fx = 6$$

за $x = 0$ добијамо $C = 6$, а за $x = -1$ добијамо $F = -6$

$$I = -\frac{1}{2x^2} + \frac{3}{x} + 6\ln x + \frac{1}{2(x+1)^3} + \frac{3}{x+1} - 6\ln(x+1) + C$$

Комплексне нуле

$$Q(x) = Q_1(x)(x^2 + bx + c), \qquad b^2 - 4ac < 0$$

где је $Q_1(x)$ као у претходном случају

Триному $x^2 + bx + c$ одговара сабирак $\dfrac{Ax + B}{x^2 + bx + c}$

$\displaystyle\int \frac{Ax + B}{x^2 + bx + c} dx$ се своди на

$$\int \frac{d(x^2 + bx + c)}{x^2 + bx + c} dx = \ln(x^2 + bx + c) + C$$

и

$$\int \frac{dx}{(x - b)^2 + a^2} = \frac{1}{a} \arctan \frac{x - b}{a} + C$$

13

$$\int \frac{3x-2}{x^2+4x+13}dx = ?$$

Решење.

$$I = \frac{3}{2}\int \frac{2x+4-4-4/3}{x^2+4x+13}dx$$

$$I = \frac{3}{2}\int \frac{2x+4}{x^2+4x+13}dx - 8\int \frac{d(x+2)}{(x+2)^2+3^2}$$

$$I = \frac{3}{2}\ln|x^2+4x+13| - \frac{8}{3}\arctan\frac{x+2}{3} + C$$

14

$$I=\int\frac{x^5-1}{x^3+x^2+x}dx=?$$

Решење.

$$\frac{x^5-1}{x^3+x^2+x}=x^2-x+\frac{x^2-1}{x^3+x^2+x}$$

$$I=\frac{x^3}{3}-\frac{x^2}{2}+J,\qquad J=\int\frac{x^2-1}{x^3+x^2+x}dx$$

$$\frac{x^2-1}{x(x^2+x+1)}=\frac{A}{x}+\frac{Bx+C}{x^2+x+1}$$

$$x^2-1=A(x^2+x+1)+(Bx+C)x=(A+B)x^2+(A+C)x+A$$

$A=1,\ \ A+B=1,\ \ A+C=0,\ \ B=2,\ \ C=1$

$$\frac{x^2-1}{x^3+x^2+x}=-\frac{1}{x}+\frac{2x+1}{x^2+x+1}$$

$$J=-\ln|x|+\ln(x^2+x+1)+C=\ln\left|\frac{x^2+x+1}{x}\right|+C$$

15 Збирка, зад.141, стр.79

$$I = \int \frac{xdx}{x^3 - 1} = ?$$

Решење.

$$\frac{x}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

За $x = 1, \quad 1 = 3A, \;\; A = 1/3; \qquad$ За $x = 0, \quad 0 = A + C\cdot(-1), \;\; C = 1/3$

За $x = -1, \quad -1 = A\cdot 1 + (-B + C)\cdot(-2), \;\; -1 = -1/3 + 2B, \;\; B = -1/3$

$$I = \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{3}\int \frac{x-1}{x^2+x+1}dx = \frac{1}{3}\ln|x-1| - \frac{1}{3}\cdot\frac{1}{2}\int \frac{2x-2}{x^2+x+1}dx$$

$$I = \frac{1}{3}\ln|x-1| - \frac{1}{6}\int \frac{d(x^2+x+1)}{x^2+x+1} - \frac{1}{2}\int \frac{dx}{(x+1/2)^2 + (\sqrt{3}/2)^2}$$

$$I = \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln(x^2+x+1) - \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\arctan\frac{x+1/2}{\sqrt{3}/2} + C$$

16

$$I = \int \frac{dx}{x(1+x^2)^2} = ?$$

Решење.

$$\frac{1}{x(1+x^2)^2} = \frac{A}{x} + \frac{Bx+C}{1+x^2} + \frac{Dx+E}{(1+x^2)^2}$$

$$1 = A(1+x^2)^2 + (Bx+C)x(1+x^2) + (Dx+e)x$$

За $x = 0$ добијамо $A = 1$

За $x = i, \quad 1 = (Di+E)i, \; 1 = -D + Ei, \; D = -1, \; E = 0$

$$1 = (1+x^2)^2 + (Bx+C)x(1+x^2) - x^2$$

За $x = 1, \quad 1 = 4 + (B+C)\cdot 2 - 1, \; B + C = -1$

За $x = -1, \quad 1 = 4 + (-B+C)\cdot(-1)\cdot 2 - 1$

$$-2 = (B-C)\cdot 2, \; B - C = -1, \; 2B = -2, \; B = -1, \; C = 0$$

$$I\int\frac{dx}{x}-\int\frac{xdx}{1+x^2}-\int\frac{xdx}{(1+x^2)^2}=\ln|x|-\frac{1}{2}\ln(1+x^2)-\frac{1}{2}\int\frac{d(1+x^2)}{(1+x^2)^2}$$

$$I=\ln\frac{|x|}{\sqrt{1+x^2}}+\frac{1}{2(1+x^2)}+C$$

M2 - ВЕЖБЕ - 10. термин

Драган Ђорић

Факултет организационих наука

2009/2010

Садржај

Интеграција тригонометријских ф.

Ако је $f(x) = R(\sin x, \cos x)$, где је $R(u, v)$ рационална функција аргумената u и v, тада се сменом $\tan\dfrac{x}{2} = t$ интеграција функције f своди на интеграцију рационалних функција.

$$x = 2\arctan t, \qquad dx = \frac{2dt}{1+t^2},$$

$$\sin x = \frac{2\sin\dfrac{x}{2}\cos\dfrac{x}{2}}{\cos^2\dfrac{x}{2} + \sin^2\dfrac{x}{2}} = \frac{2t}{1+t^2},$$

$$\cos x = \frac{\cos^2\dfrac{x}{2} - \sin^2\dfrac{x}{2}}{\cos^2\dfrac{x}{2} + \sin^2\dfrac{x}{2}} = \frac{1-t^2}{1+t^2}$$

$$\int R(\sin x, \cos x)dx = \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2dt}{1+t^2}$$

1 Збирка, зад.150, стр.80

$$I = \int \frac{dx}{1 + \sin x + \cos x}$$

Решење.

$$\tan \frac{x}{2} = t, \quad dx = \frac{2dt}{1+t^2}, \quad \sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$$

$$I = \int \frac{1}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2} = \int \frac{2dt}{2+2t} = \int \frac{dt}{1+t}$$

$$I = \ln|1+t| + C = \ln\left(1 + \tan\frac{x}{2}\right) + C$$

2 Збирка, зад.154, стр.80

$$\int \frac{1-\sin x+\cos x}{1+\sin x-\cos x}dx$$

Решење.

$$\tan\frac{x}{2}=t,\quad dx=\frac{2dt}{1+t^2},\quad \sin x=\frac{2t}{1+t^2},\quad \cos x=\frac{1-t^2}{1+t^2}$$

$$f(x)=\frac{1-\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}{1+\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^1}}=\frac{1+t^2-2t+1-t^2}{1+t^2+2t-1+t^2}=\frac{1-t}{t(1+t)}$$

$$I=2\int\frac{(1-t)dt}{t(t+1)(t^2+1)}$$

$$\frac{1-t}{t(t+1)(t^2+1)}=\frac{A}{t}+\frac{B}{t+1}+\frac{Ct+D}{t^2+1},\qquad A=1,C=0,B=D=-1$$

$$I=2\int\frac{dt}{t}-2\int\frac{dt}{t+1}-2\int\frac{dt}{t^2+1}=2\ln\frac{t}{t+1}-2\arctan t+C$$

$$I=\ln\frac{\tan\frac{x}{2}}{\tan\frac{x}{2}+1}-x+C$$

Триг. ф. - Специјални случајеви

$R(u,-v) = -R(u,v), \quad \sin x = t$

$$\int \frac{\cos x dx}{\sin^2 x - 3\sin x + 2} = \int \frac{dt}{t^2 - 3t + 2}$$

$R(-u,v) = -R(u,v), \quad \cos x = t$

$$\int \frac{\sin x(1-\cos x)}{\cos x(1+\cos^2 x)} dx = \int \frac{t-1}{t(1+t^2)} dt$$

$R(-u,-v) = R(u,v), \quad \tan x = t$

$$\int \frac{\sin^3 x}{\cos^5 x} dx = \int t^3 dt = \frac{t^4}{4} + C = \frac{1}{4}\tan^3 x + C$$

3

$$I = \int \frac{dx}{4 - 3\cos^2 x + 5\sin^2 x}$$

Решење.

$\tan x = t$

$$I = \int \frac{dx}{\cos^2 x + 9\sin^2 x} = \int \frac{dt}{1 + (3t)^2} = \frac{1}{3}\arctan(3\tan x) + C$$

4

$I = \int \frac{1-\tan x}{1+\tan x} dx$

Решење.

$\tan x = t$

$$I = \int \frac{1-t}{(1+t)(1+t^2)} dt$$

Међутим, може и без смене,

$$\begin{aligned} I &= \int \frac{\cos x - \sin x}{\sin x + \cos x} dx \\ &= \int \frac{d(\sin x + \cos x)}{\sin x + \cos x} \\ &= \ln|\sin x + \cos x| + C. \end{aligned}$$

5 Збирка, зад.161, стр.80

$$I = \int \frac{dx}{1+\sin^2 x}$$

Решење.

$\tan x = t$, $\tan^2 t = t^2$, $\dfrac{1-\cos^2 t}{\cos^2 t} = t^2$, $\dfrac{1}{\cos^2 t} = 1 + t^2$

$$I = \int \frac{dx/\cos^2 x}{1/\cos^2 x + \tan^2 x} = \int \frac{dt}{t^2+1+t^2} = \int \frac{dt}{1+2t^2}$$

$$I = \frac{1}{\sqrt{2}} \int \frac{d(\sqrt{2}t)}{1+(\sqrt{2}t)^2} = \frac{1}{\sqrt{2}} \cdot \arctan(\sqrt{2}t) + C$$

$$I = \frac{1}{\sqrt{2}} \arctan(\sqrt{2}\tan x) + C$$

Може и $\dfrac{1}{1+\sin^2 x} = \dfrac{1}{\cos^2 x + 2\sin^2 x} = \dfrac{1}{\cos^2 x} \cdot \dfrac{1}{1+2\tan^2 x}$

Интеграција ирационалних функција

1. Функције облика $R(x, \sqrt{ax+b})$, $\quad \sqrt{ax+b} = t$

2. Функције облика $R\left(x, \sqrt[n]{\dfrac{ax+b}{cx+d}}\right)$, $\quad \sqrt[n]{\dfrac{ax+b}{cx+d}} = t$

3. Функције облика $R(x, \sqrt{a^2 - x^2})$, $\quad x = a \sin t$

4. Функције облика $R(x, \sqrt{x^2 - a^2})$, $\quad x = a \cosh t$ или $x = \dfrac{a}{\sin t}$

5. Функције облика $R(x, \sqrt{x^2 + a^2})$, $\quad x = a \sinh t$ или $x = a \cdot \tan t$

6. Функције облика $R(x, \sqrt{ax^2 + bx + c})$ - своди се на 3.,4. или 5., а могу и *Ојлерове смене*:

$\sqrt{ax^2 + bx + c} = t - x\sqrt{a}$ за $a > 0$

$\sqrt{ax^2 + bx + c} = xt - \sqrt{c}$ за $c > 0$

$\sqrt{ax^2 + bx + c} = (x - \alpha)t$ за $ax^2 + bx + c = a(x - \alpha)(x - \beta)$

Користе се таблични интеграли:

1. $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}} = \ln(x+\sqrt{x^2+a^2}) + C$

2. $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \ln(x+\sqrt{x^2-a^2}) + C$

3. $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C$

а такође и интеграли:

4. $\displaystyle\int \sqrt{x^2+a^2}dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln(x+\sqrt{x^2+a^2}) + C$

5. $\displaystyle\int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln|x+\sqrt{x^2-a^2}| + C$

6. $\displaystyle\int \sqrt{a^2-x^2}dx = \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + C$

$$I = \int \frac{dx}{(x+1)\sqrt{1-x}}$$

Решење.

$\sqrt{1-x} = t,\ x = 1 - t^2,\ dx = -2tdt,\ x + 1 = -t^2 + 2$

$$I = 2\int \frac{dt}{t^2-2} = \frac{1}{\sqrt{2}} \ln\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right| + C = \frac{1}{\sqrt{2}} \ln\left|\frac{\sqrt{1-x}-\sqrt{2}}{\sqrt{1-x}+\sqrt{2}}\right| + C$$

7

$$I = \int \frac{1}{(1+x)^2} \sqrt[3]{\frac{1-x}{1+x}} dx$$

Решење.

$$\sqrt[3]{\frac{1-x}{1+x}} = t$$

$$x = \frac{1-t^3}{1+t^3}, \qquad dx = -\frac{6t^2}{(1+t^3)^2}, \qquad 1+x = \frac{2}{1+t^3}$$

$$I = \frac{3}{2}\int t^3 dt = \frac{3}{8}t^4 + C = \frac{3}{8}\left(\frac{1-x}{1+x}\right)^{4/3} + C$$

8 Збирка, зад.167, стр.80

$$I=\int\frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}dx$$

Решење.

$$\sqrt{\frac{x+1}{x-1}}=t,\quad \frac{x+1}{x-1}=t^2,\quad x=\frac{1+t^2}{t^2-1},\quad dx=-\frac{4tdt}{(t^2-1)^2}$$

$$I=\int\frac{\sqrt{\frac{x+1}{x-1}}-1}{\sqrt{\frac{x+1}{x-1}}+1}dx=-4\int\frac{tdt}{(t-1)(t+1)^3}$$

$$\frac{t}{(t-1)(t+1)^3}=\frac{1}{8}\cdot\frac{1}{t-1}-\frac{1}{8}\cdot\frac{1}{t+1}-\frac{1}{4}\cdot\frac{1}{(t+1)^2}+\frac{1}{2}\cdot\frac{1}{(t+1)^3}$$

$$I=-\frac{1}{2}\ln|t-1|+\frac{1}{2}\ln|t+1|-\frac{1}{t+1}+\frac{1}{(t+1)^2}+C$$

$$I=\frac{1}{2}\ln\left|\frac{t+1}{t-1}\right|-\frac{t}{(t+1)^2}+C,\qquad t=\sqrt{\frac{x+1}{x-1}}$$

9 Збирка, зад.169, стр.80

$$\int \frac{dx}{(1-x)\sqrt{1-x^2}}$$

Решење.

$$(1-x)\sqrt{1-x^2} = (1-x)\sqrt{(1-x)(1+x)} = (1-x)(1+x)\sqrt{\frac{1-x}{1+x}}$$

$$\frac{1-x}{1+x} = t^2, \quad x = \frac{1-t^2}{1+t^2}, \;\; dx = \frac{-4tdt}{(1+t^2)^2}, 1-x = \frac{2t^2}{1+t^2}, \;\; 1+x = \frac{2}{1+t^2}$$

$$I = -\int \frac{4tdt}{(1+t^2)^2 \cdot \frac{2t^2}{1+t^2} \cdot \frac{2}{1+t^2} \cdot t} = -\int \frac{dt}{t^2} = \frac{1}{t} + C$$

$$I = \sqrt{\frac{1+x}{1-x}} + C$$

Друго решење.

$x = \sin t$

$$\begin{aligned}
I &= \int \frac{dt}{1-\sin t} = \int \frac{1+\sin t}{1-\sin^2 t} dt \\
&= \int \frac{1+\sin t}{\cos^2 t} dt = \tan t - \int \frac{d(\cos t)}{\cos^2 t} \\
&= \tan t + \frac{1}{\cos t} + C = \frac{\sin t + 1}{\cos t} + C \\
&= \frac{\sin t + 1}{\sqrt{1-\sin^2 t}} = \frac{x+1}{\sqrt{1-x^2}} + C \\
&= \sqrt{\frac{1+x}{1-x}} + C
\end{aligned}$$

10

$$I = \int \frac{dx}{\sqrt{1-x^2}-1}$$

Решење.

$x = \sin t$

$$\begin{aligned} I &= \int \frac{\cos t dt}{\cos t - 1} \\ &= \int dt - \int \frac{dt}{1-\cos t} \\ &= t - \int \frac{1+\cos t}{\sin^2 t} dt \\ &= t + \cot t + \frac{1}{\sin t} + C \\ &= \arcsin x + \frac{\sqrt{1-x^2}}{x} + \frac{1}{x} + C. \end{aligned}$$

11

$$I = \int \sqrt{(x^2-1)^3}dx$$

Решење.

$x = \cosh t$

$$\begin{aligned} I &= \int \sqrt{(\cosh^2 t - 1)^3} \sinh t dt \\ &= \int \sinh^4 t dt \\ &= \frac{1}{4}\int (\cosh 2t - 1)^2 dt \\ &= \frac{1}{4}\int \cosh^2 t dt - \frac{1}{2}\int \cosh 2t dt + \frac{1}{4}\int dt \\ &= \frac{1}{32}\sinh 4t - \frac{1}{4}\sinh 2t + \frac{3}{8}t + C. \end{aligned}$$

Како је

$$t = arcosh\ x = \ln(x + \sqrt{x^2 - 1}),$$

$$\sinh 2t = 2 \sinh t \cosh t = 2x\sqrt{x^2 - 1},$$

$$\sinh 4t = 2 \sinh 2t \cosh 2t = 4x\sqrt{x^2 - 1}(2x^2 - 1),$$

имамо да је

$$\begin{aligned} I &= \frac{1}{8}x(2x^2 - 1)\sqrt{x^2 - 1} - \frac{1}{2}x\sqrt{x^2 - 1} + \\ &\quad \frac{3}{8}\ln(x + \sqrt{x^2 - 1}) + C. \end{aligned}$$

12

$$I = \int \frac{dx}{x^2\sqrt{x^2+1}}$$

Решење.

$x = \sinh t$

$$\begin{aligned} I &= \int \frac{\cosh t dt}{\sinh^2 t \cosh t} \\ &= \int \frac{dt}{\sinh^2 t} \\ &= -\coth t + C \\ &= -\frac{\sqrt{1+\sinh^2 t}}{\sinh t} + C \\ &= -\frac{\sqrt{1+x^2}}{x} + C. \end{aligned}$$

13

$$I = \int \frac{\sqrt{x^2+1}}{x^4}dx$$

Решење.

$x = \tan t$

$$I = \int \frac{\cos t}{\sin^4 t}dt = -\frac{1}{3\sin^3 t} + C$$

Како је

$$\sin^2 t = \frac{\sin^2 t}{\sin^2 t + \cos^2 t} = \frac{\tan^2 t}{1+\tan^2 t} = \frac{x^2}{1+x^2},$$

то је $\sin t = \dfrac{x}{\sqrt{1+x^2}}$, па је

$$I = -\frac{1}{3}\frac{\sqrt{(1+x^2)^3}}{x^3} + C$$

14

$$I = \int \sqrt{x^2 + 6x}\, dx$$

Решење.

$$I = \int \sqrt{(x+3)^2 - 3^2}\, d(x+3)$$

$$\int \sqrt{t^2 - a^2} dt = \frac{t}{2}\sqrt{t^2 - a^2} - \frac{a^2}{2} \ln|t + \sqrt{t^2 - a^2}| + C$$

$$I = \frac{x+3}{2}\sqrt{x^2 + 6x} - \frac{9}{2} \ln|x + 3 + \sqrt{x^2 + 6x}| + C$$

15 Збирка, зад.172, стр.80

$$I = \int \frac{5x+4}{\sqrt{x^2+2x+5}}dx$$

Решење.

$$\begin{aligned}
I &= \frac{5}{2}\int \frac{2x+8/5}{\sqrt{x^2+2x+5}}dx \\
&= \frac{5}{2}\int \frac{2x+2}{\sqrt{x^2+2x+5}}dx - \frac{5}{2}\cdot\frac{2}{5}\int \frac{dx}{\sqrt{x^2+2x+5}} \\
&= 5\int \frac{d(x^2+2x+5)}{2\sqrt{x^2+2x+5}}dx - \int \frac{dx}{\sqrt{(x+1)^2+2^2}} \\
&= 5\sqrt{x^2+2x+5} - \ln(x+1+\sqrt{x^2+2x+5}) + C
\end{aligned}$$

16

$$I = \int \frac{x dx}{\sqrt{x^2 + x + 1}}$$

Решење.

$$\begin{aligned} I &= \frac{1}{2}\int \frac{2x+1}{\sqrt{x^2+x+1}}dx - \frac{1}{2}\int \frac{dx}{\sqrt{x^2+x+1}} \\ &= \sqrt{x^2+x+1} - \frac{1}{2}\ln\left(x + \frac{1}{2} + \sqrt{x^2+x+1}\right). \end{aligned}$$

17 Збирка, зад.171, стр.80

$$I = \int \frac{x^2dx}{\sqrt{x^2+x+1}}$$

Решење.

$$\begin{aligned}
I &= \int \frac{x^2+x+1}{\sqrt{x^2+x+1}}dx - \int \frac{x+1/2}{\sqrt{x^2+x+1}}dx - \frac{1}{2}\int \frac{dx}{\sqrt{x^2+x+1}} \\
&= \int \sqrt{(x+1/2)^2+3/4}dx - \int \frac{2x+1}{2\sqrt{x^2+x+1}}dx - \frac{1}{2}\int \frac{dx}{\sqrt{(x+1/2)^2+3/4}} \\
&= \frac{x+1/2}{2}\sqrt{x^2+x+1} + \frac{3}{8}\ln(x+1/2+\sqrt{x^2+x+1}) \\
&\quad -\sqrt{x^2+x+1} - \frac{1}{2}\ln(x+1/2+\sqrt{x^2+x+1}) + C \\
&= \left(\frac{2x+1}{4}-1\right)\sqrt{x^2+x+1} + \left(\frac{3}{8}-\frac{1}{2}\right)\ln(x+1/2+\sqrt{x^2+x+1}) + C \\
&= \frac{2x-3}{4}\sqrt{x^2+x+1} - \frac{1}{8}\ln(x+1/2+\sqrt{x^2+x+1}) + C
\end{aligned}$$

18 Збирка, зад.174, стр.81

$$\int \sqrt{4x^2 - 4x + 3}\, dx$$

Решење.

$$I = \int \sqrt{(2x-1)^2 + 2}\, dx, \qquad 2x - 1 = t, \ \ 2dx = dt$$

$$\begin{aligned} I &= \frac{1}{2} \int \sqrt{t^2 + 2}\, dt \\ &= \frac{1}{2} \left(\frac{t}{2} \sqrt{t^2 + 2} + \frac{2}{2} \ln(t + \sqrt{t^2 + 2}) \right) + C \\ &= \frac{2x - 1}{4} \sqrt{4x^2 - 4x + 3} + \frac{1}{2} \ln(2x - 1 + \sqrt{4x^2 - 4x + 3}) + C \end{aligned}$$

19

$$I=\int x^2\sqrt{4-x^2}\,dx$$

Решење.

$x=2\sin t$

$$I=\int 4\sin^2 t\cdot 2\cos t\cdot 2\cos t dt=16\int \sin^2 t\cos^2 t dt=16J$$

$$\begin{aligned}\sin^2 t\cos^2 t &= \frac{1-\cos 2t}{2}\cdot\frac{1+\cos 2t}{2}\\ &= \frac{1}{4}(1-\cos^2 2t)=\frac{1}{4}\left(1-\frac{1+\cos 4t}{2}\right)=\frac{1}{8}-\frac{1}{8}\cos 4t\end{aligned}$$

$$I=16J=16\left(\frac{1}{8}t-\frac{1}{32}\sin 4t\right)+C=2t-\frac{1}{2}\sin 4t+C$$

Како је

$$\begin{aligned}\sin 4t &= 2\sin 2t\cos 2t \\ &= 4\sin t\cdot\cos t\cdot(\cos^2 t-\sin^2 t) \\ &= 4\sin t\sqrt{1-\sin^2 t}(1-2\sin t) \\ &= 4\cdot\frac{x}{2}\sqrt{1-\frac{x^2}{4}}\left(1-2\cdot\frac{x^2}{4}\right) \\ &= x\sqrt{4-x^2}\left(1-\frac{x^2}{2}\right) \\ &= \frac{x}{2}\sqrt{4-x^2}(2-x^2)\end{aligned}$$

то је

$$I = 2\arcsin\frac{x}{2}-\frac{x}{4}\sqrt{4-x^2}(2-x^2)+C$$

Друго решење.

$$u = x, \;\; dv = x\sqrt{4-x^2}dx, \qquad du = dx, \;\; v = -\frac{1}{3}(4-x^2)\sqrt{4-x^2}$$

$$I = -\frac{x}{3}(4-x^2)^{3/2} + \frac{1}{3}(4-x^2)\sqrt{4-x^2}dx$$

$$I = -\frac{x}{3}(4-x^2)^{3/2} + \frac{4}{3}\int\sqrt{4-x^2} - \frac{1}{3}I$$

$$\frac{4}{3}I = -\frac{x}{3}(4-x^2)^{3/2} + \frac{4}{3}J, \qquad J = \int\sqrt{4-x^2}dx$$

$$J = \int 2^2 - x^2dx = 2\arcsin\frac{x}{2} + \frac{x}{2}\sqrt{4-x^2} + C$$

$$I = -x(4-x^2)^{3/2} + 8\arcsin\frac{x}{2} + 2x\sqrt{4-x^2} + C$$

$$I = 2\arcsin\frac{x}{2} + x\sqrt{4-x^2}\left(\frac{1}{2} - 1 + \frac{x^2}{4}\right) + C$$

$$I = 2\arcsin\frac{x}{2} + \frac{x}{4}\sqrt{4-x^2}(x^2-2) + C$$

20 Збирка, зад.170, стр.80

$$I = \int \frac{dx}{(1+x)\sqrt{1+x-x^2}}$$

Решење. Ојлерова смена $\sqrt{1+x-x^2} = tx - 1$

$$1+x-x^2 = t^2x^2 - 2tx + 1, \;\; x(1-x) = x(t^2x - 2t), \;\; 1 - x = t^2x - 2t$$

$$x = \frac{1+2t}{t^2+1}, \;\; dx = -2 \cdot \frac{t^2+t-1}{(t^2+1)^2} dt$$

$$\sqrt{1+x-x^2} = tx - 1 = t \cdot \frac{1+2t}{t^2+1} = \frac{t^2+t-1}{t^2+1}$$

$$1 + x = \frac{t^2+2t+2}{t^2+1}$$

$$I = -2 \int \frac{t^2+1}{t^2+2t+2} \cdot \frac{t^2+1}{t^2+t-1} \cdot \frac{t^2+t-1}{(t^2+1)^2} dt = -2 \int \frac{dt}{1+(t+1)^2}$$

$$I = -2\arctan(t+1) + C = -2\arctan\frac{1+x+\sqrt{1+x-x^2}}{x} + C$$

М2 - ВЕЖБЕ - 11. термин

Драган Ђорић

Факултет организационих наука

2009/2010

Садржај

ОДРЕЂЕНИ ИНТЕГРАЛ

1. Њутн Лајбницова формула
2. Метода смене
3. Метода парцијалне интеграције
4. Примена (површина равних фигура)

Услови интеграбилности

Неопходан услов интеграбилности

$f \in \mathcal{R}[a,b] \quad \Rightarrow \quad f$ ограничена на $[a,b]$

Довољни услови интеграбилности

1. $f \in C[a,b] \quad \Rightarrow \quad f \in \mathcal{R}[a,b]$

2. f ограничена и има коначан број прекида на $[a,b] \quad \Rightarrow \quad f \in \mathcal{R}[a,b]$

На пример, за $f(x) = \dfrac{1}{x}$ и $g(x) = \dfrac{x}{\sin x}$ имамо

$f \in \mathcal{R}[1,2], \quad f \notin \mathcal{R}[-1,1], \quad g \in \mathcal{R}[-\pi/2, \pi/2], \quad g \notin [\pi/2, 3\pi/2]$

Њутн Лајбницова формула

Ако је $f \in C[a,b]$ и ако је F примитивна за f на $[a,b]$, тада је

$$\int_a^b f(x)dx = F(b) - F(a) = F(x)\Big|_a^b$$

На пример, (Збирка - зад.32, стр.84 и зад.40, стр.85)

1. $\displaystyle \int_e^{e^2} \frac{dx}{x\ln x} = \int_e^{e^2} \frac{d(\ln x)}{\ln x} = \ln(\ln x)\Big|_e^{e^2} = \ln(\ln e^2) - \ln(\ln e) = \ln 2$

2. $\displaystyle \int_1^{e} \frac{dx}{x(1+\ln^2 x)} = \int_1^{e} \frac{d(\ln x)}{1+\ln^2 x} = \arctan(\ln x)\Big|_1^{e}$

$$= \arctan(\ln e) - \arctan(\ln 1) = \frac{\pi}{4}$$

Ако је F примитивна за f на (a,b), тада је

$$\int_a^b f(x)dx = F(b_-) - F(a_+) = F(x)\Big|_{a_+}^{b_-}$$

1 Збирка, зад.41, стр.85

$$I = \int_0^1 \sqrt{4-x^2}dx$$

Решење.

$$\int \sqrt{a^2-x^2}dx = \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + C$$

$$F(x) = \frac{2^2}{2}\arcsin\frac{x}{2} + \frac{x}{2}\sqrt{4-x^2}$$

$$I = F(x)\Big|_0^1 = 2\arcsin\frac{1}{2} + \frac{1}{2}\sqrt{4-1} = 2\cdot\frac{\pi}{3} + \frac{\sqrt{3}}{2}$$

$$I = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$

2 Збирка, зад.45, стр.85

$$I = \int_0^{\pi/4} \frac{dx}{1 + 2\sin^2 x}$$

Решење.

$$I = \int_0^{\pi/4} \frac{dx}{\cos^2 x + 3\sin^2 x} = \int_0^{\pi/4} \frac{d(\tan x)}{1 + 3\tan^2 x}$$

$$I = \frac{1}{\sqrt{3}} \int_0^{\pi/4} \frac{d(\sqrt{3}\tan x)}{1 + (\sqrt{3}\tan x)^2}$$

$$I = \frac{1}{\sqrt{3}} \arctan(\sqrt{3}\tan x)\Big|_0^{\pi/4}$$

$$I = \frac{1}{\sqrt{3}} \arctan\sqrt{3} = \frac{\pi}{3\sqrt{3}}$$

3

$$I = \int_{1/e}^{e} |\ln x| dx$$

Решење.

$$I = -\int_{1/e}^{1} \ln x dx + \int_{1}^{e} \ln x dx$$

$$\int \ln x dx = x \ln x - x + C$$

$$I = -(x \ln x - x) \Big|_{1/e}^{1} + x \ln x - x \Big|_{1}^{e}$$

$$I = -\left(0 - 1 - \frac{1}{e} \ln \frac{1}{e} + \frac{1}{e}\right) + e - e - (0 - 1)$$

$$I = 1 - \frac{2}{e} + 1 = 2 - \frac{2}{e}$$

$$I = \int_0^{\pi} \frac{dx}{3 + 2\cos x}$$

Решење.

$$F(x) = \frac{2}{\sqrt{5}} \arctan\left(\frac{1}{\sqrt{5}} \tan\frac{x}{2}\right), \qquad x \in [0, \pi)$$

$$I = F(x)\Big|_0^{\pi_-} = F(\pi_-) - F(0)$$

$$I = \lim_{x \to \pi_-} \frac{2}{\sqrt{5}} \arctan\left(\frac{1}{\sqrt{5}} \tan\frac{x}{2}\right) - 0$$

$$I = \frac{\pi}{\sqrt{5}}$$

Метода смене

$f \in C[a,b], \quad \varphi \in C^{(1)}[\alpha,\beta], \quad \varphi(\alpha) = a,\ \varphi(\beta) = b$

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt$$

- Ако je f непарна, тада je $\int_{-a}^{a} f(x)dx = 0$

На пример, $\int_{-1/2}^{1/2} \cos x \ln \frac{1+x}{1-x} dx = 0$

- Ако je f парна, тада je $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

5

$$I = \int_0^a x^2 \sqrt{a^2 - x^2}\, dx, a > 0$$

Решење.

$x = a\sin t, \qquad dx = a\cos t dt,$

x	0	a
t	0	$\pi/2$

$$I = \int_0^{\pi/2} a^2 \sin^2 t \sqrt{a^2 - a^2\sin^2 t}\, a\cos t dt = a^4 \int_0^{\pi/2} \sin^2 t \cos^2 t dt$$

$$I = \frac{a^4}{4}\int_0^{\pi/2} \sin^2 2t dt = \frac{a^4}{8}\int_0^{\pi/2} (1 - \cos 4t) dt$$

$$I = \frac{a^4}{8}\left(t - \frac{1}{4}\sin 4t\right)\Big|_0^{\pi/2}$$

$$I = \frac{\pi}{16} a^4$$

6

$$I=\int_1^{\sqrt{3}}\frac{x^3+1}{x^2\sqrt{4-x^2}}dx$$

Решење.

$x=2\sin t,\qquad dx=2\cos t dt,$

x	1	$\sqrt{3}$
t	$\pi/6$	$\pi/3$

$$\int_{\pi/6}^{\pi/3}\frac{(8\sin^3 t+1)2\cos t}{4\sin^2 t\sqrt{4-4\sin^2 t}}dt$$

$$I=2\int_{\pi/6}^{\pi/3}\sin t dt+\frac{1}{4}\int_{\pi/6}^{\pi/3}\frac{dt}{\sin^2 t}$$

$$I=-2\cos t\Big|_{\pi/6}^{\pi/3}-\frac{1}{4}\cos t\Big|_{\pi/6}^{\pi/3}=-2\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\right)-\frac{1}{4}\left(\frac{\sqrt{3}}{3}-\sqrt{3}\right)$$

$$I=\frac{7}{6}\sqrt{3}-1$$

7

$$I = \int_0^{\sqrt{2}} x^3 \sqrt{16 - x^8}\, dx$$

Решење.

$$x^4 = t, \;\; x^3 dx = \frac{1}{4} dt$$

$$I = \frac{1}{4} \int_0^4 \sqrt{16 - t^2}\, dt$$

$$t = 4 \sin u$$

$$I = 4 \int_0^{\pi/2} |\cos u| \cdot \cos u \;u = 4 \int_0^{\pi/2} \cos^2 u du$$

$$I = 2 \in_0^{\pi/2} (1 + \cos 2u) du = 2u + \sin 2u \,\Big|_0^{\pi/2}$$

$$I = \pi$$

$$I = \int_4^5 x\sqrt{x^2 - 4x}\, dx$$

Решење.

$$I = \int_4^5 x\sqrt{(x-2)^2 - 4}\, dx$$

$$x - 2 = t, \quad dx = dt$$

$$I = \int_2^3 (2+t)\sqrt{t^2 - 4}\, dt$$

$$I = 2\int_2^3 \sqrt{t^2 - 4}\, dt + \int_2^3 t\sqrt{t^2 - 4}\, dt$$

$$I = 2\left(\frac{1}{2}\sqrt{t^2 - 4} - 2\ln\left|t + \sqrt{t^2 - 4}\right|\right)\Big|_2^3 + \frac{1}{3}(t^2 - 4)^{3/2}\Big|_2^3$$

$$I = \frac{14}{3}\sqrt{5} - 4\ln\frac{3+\sqrt{5}}{2}$$

9

$$I = \int_0^{\pi/2} \frac{dx}{1+\sin x+\cos x}$$

Решење.

$$\tan\frac{x}{2} = t, \qquad \begin{array}{c||c|c} x & 0 & \pi/2 \\ \hline t & 0 & 1 \end{array}, \qquad \sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$$

$$I = \int_0^1 \frac{1}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2}$$

$$I = \int_0^1 \frac{2dt}{1+t^2+2t+1-t^2} = \int_0^1 \frac{2dt}{2+2t} = \int_0^1 \frac{dt}{1+t}$$

$$I = \ln(1+t)\Big|_0^1 = \ln 2$$

10 Колоквијум, 2002.

$$\int_0^{\pi/2} \frac{\sin^3 x}{\sin^3 x + \cos^3 x} dx$$

Решење.

$$x = \frac{\pi}{2} - t, \qquad dx = -dt, \qquad \begin{array}{c||c|c} x & 0 & \pi/2 \\ \hline t & \pi/2 & 0 \end{array}$$

$$I = \int_{\pi/2}^{0} \frac{\cos^3 t}{\sin^3 t + \cos^3 t}(-dt) = \int_0^{\pi/2} \frac{\cos^3 t}{\sin^3 t + \cos^3 t} dt$$

$$2I = \int_0^{\pi/2} \frac{\sin^3 x + \cos^3 x}{\sin^3 x + \cos^3 x} dx = \int_0^{\pi/2} dt = \frac{\pi}{2}$$

$$I = \pi/4$$

11

$$I = \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$$

Решење.

$x = \pi - t, \quad dx = -dt,$

x	0	π
t	π	0

$$I = \pi \int_0^{\pi} \frac{\sin x dx}{1 + \cos^2 x} - \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$$

$$I = \pi J - I, \quad 2I = \pi J, \quad I = \frac{\pi}{2} J$$

$$J = -\int_0^{\pi} \frac{d(\cos x)}{1 + \cos^2 x} = -\arctan(\cos x)\Big|_0^{\pi} = -(\arctan(-1) - \arctan 1) = \frac{\pi}{2}$$

$$I = \frac{\pi^2}{4}$$

12

$$I = \int_{\pi/4}^{\pi/3} \frac{dx}{\sin x \cos^3 x}$$

Решење.

$\cos x = t$

$$I = \int_{\pi/4}^{\pi/3} \frac{\sin x dx}{(1 - \cos^2 x)\cos^3 x} = \int_{1/2}^{\sqrt{2}/2} \frac{dt}{(1 - t^2)t^3}$$

$$\frac{1}{(1 - t^2)t^3} = \frac{1}{2} \cdot \frac{1}{1 - t} - \frac{1}{2} \cdot \frac{1}{1 + t} + \frac{1}{t} + \frac{1}{t^3}$$

$$I = \frac{1}{2} \int_{1/2}^{\sqrt{2}/2} \frac{dt}{1 - t} - \frac{1}{2} \int_{1/2}^{\sqrt{2}/2} \frac{dt}{1 + t} + \int_{1/2}^{\sqrt{2}/2} \frac{dt}{t} + \int_{1/2}^{\sqrt{2}/2} \frac{dt}{t^3}$$

$$I = -\frac{1}{2} \ln(t^2 - 1) + \ln t - \frac{1}{2} \cdot \frac{1}{t^2} \Big|_{1/2}^{\sqrt{2}/2}$$

$$I = \frac{1}{2} \ln 3 + 1$$

Метода парцијалне интеграције

$$u, v \in C^{(1)}[a,b] \quad \Rightarrow \quad \int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

13

$$I=\int_0^{1/2} x\ln\frac{1+x}{1-x}dx$$

Решење.

$$u=\ln\frac{1+x}{1-x},\quad dv=xdx \qquad du=\frac{2dx}{1-x^2},\quad v=\frac{x^2}{2}$$

$$I=\frac{x^2}{2}\ln\frac{1+x}{1-x}\Big|_1^{1/2}-\int_0^{1/2}\frac{x^2dx}{1-x^2}$$

$$I=\frac{1}{8}\ln 3-\int_0^{1/2}\frac{x^2-1+1}{1-x^2}dx$$

$$I=\frac{1}{8}\ln 3+x\Big|_0^{1/2}-\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|\Big|_0^{1/2}$$

$$I=\frac{1}{2}-\frac{3}{8}\ln 3$$

14 Збирка, зад.74, стр.86

$$I = \int_0^e x^2 \ln^2 x dx$$

Решење.

$u = \ln^2 x dx,\ \ dv = x^2 dx, \qquad du = \frac{2\ln x}{x} dx,\ \ v = \frac{x^3}{3}$

$$I = \frac{x^3}{3} \ln^2 x \Big|_0^e - \frac{2}{3} \int_9^e x^2 \ln x dx = \frac{e^3}{3} - \frac{2}{3} J$$

За J $u = \ln x,\ \ dv = x^2 dx, \qquad du = \frac{dx}{x},\ \ v = \frac{x^3}{3}$

$$J = \frac{x^3}{3} \ln x \Big|_0^e - \frac{1}{3} \int_0^e x^2 dx = \frac{e^3}{3} - \frac{1}{3} \cdot \frac{x^3}{3} \Big|_0^e = \frac{2}{9} e^3$$

$$I = \frac{e^3}{3} - \frac{4}{27} e^3 = \frac{5}{27} e^3$$

15 Збирка, зад.80, стр.87

$$I = \int_0^3 \arcsin\sqrt{\frac{x}{x+1}}dx$$

Решење.

$$u = \arcsin\sqrt{\frac{x}{x+1}}, \qquad dv = dx$$

$$du = \frac{1}{\sqrt{1-\frac{x}{x+1}}} \cdot \frac{1}{2\sqrt{\frac{x}{x+1}}} \cdot \frac{x+1-x}{(x+1)^2}dx = \frac{1}{2\sqrt{x}} \cdot \frac{dx}{x+1}, \quad v = x$$

$$I = x\arcsin\sqrt{\frac{x}{x+1}}\,\Big|_0^3 - \int_0^3 \frac{xdx}{2\sqrt{x}(x+1)} = 3\arcsin\sqrt{\frac{3}{4}} - \int_0^3 \frac{\sqrt{x}^2 d(\sqrt{x})}{1+\sqrt{x}^2}$$

$$I = 3\arcsin\frac{\sqrt{3}}{2} - (\sqrt{x} - \arctan\sqrt{x})\,\Big|_0^3$$

$$I = 3 \cdot \frac{\pi}{3} - \sqrt{3} + \frac{\pi}{3} = \frac{4}{3}\pi - \sqrt{3}$$

$$\int_0^1 \frac{x\ln(x+\sqrt{1+x^2})}{(1+x^2)^2}dx$$

$$u=\ln(x+\sqrt{1+x^2}),\ \ dv=\frac{xdx}{(1+x^2)^2}, \qquad du=\frac{dx}{\sqrt{1+x^2}},\ \ v=\frac{-1}{2(1+x^2)}$$

$$I=-\frac{\ln(x+\sqrt{1+x^2})}{2(1+x^2)}\Big|_0^1+\frac{1}{2}\int_0^1(1+x^2)^{-3/2}dx=-\frac{1}{4}\ln(1+\sqrt{2})+\frac{1}{2}J$$

За $J:\qquad x=tant,\ \ t\in[0,\pi/4]$

$$J=\int_0^{\pi/4}(1+\tan^2 t)^{-3/2}\cos^{-2}tdt=\int_0^{\pi/4}\cos tdt=\frac{\sqrt{2}}{4}$$

$$I=\frac{\sqrt{2}}{4}-\frac{1}{4}\ln(1+\sqrt{2})$$

17 Збирка, зад.83, стр.87

$$I_n = \int_0^a (a^2 - x^2)^n dx, \;\; a > 0$$

Решење.

$$I_n = \int_0^a (a^2 - x^2)^{n-1}(a^2 - x^2) dx = a^2 I_{n-1} - J$$

За J : $\quad u = x, \;\; x(a^2 - x^2)^{n-1} dx = dv \qquad du = dx, \;\; v = -\frac{1}{2n}(a^2 - x^2)^n$

$$J = -\frac{x}{2n}(a^2 - x^2)^n \Big|_0^a + \frac{1}{2n}\int_0^a (a^2 - x^2)^n dx = \frac{1}{2n} I_n$$

$$I_n = a^2 I_{n-1} - \frac{1}{2n} I_n, \qquad I_n = \frac{2n}{2n+1} a^2 I_{n-1}$$

$$I_n = \frac{2n}{2n+1} a^2 I_{n-1} = \frac{2n}{2n+1} \cdot \frac{2(n-1)}{2(n-1)+1} a^4 I_{n-2} = a^{2n} \frac{(2n)!!}{(2n+1)!!} I_0$$

$$I_0 = \int_0^a dx = a, \qquad I_n = a^{2n+1} \frac{(2n)!!}{(2n+1)!!}$$

Површина равне фигуре

За фигуру ограничену са $y = f(x),\ y = 0,\ x = a,\ x = b$ је

$$P = \int_a^b |f(x)| dx$$

На пример, (зад.85, стр.87)

$$y = \sin x,\ 0 \le x \le \pi \qquad P = \int_0^\pi \sin x dx = -\cos x \Big|_0^\pi = -(-1-1) = 2$$

За фигуру ограничену са $y = f(x),\ y = g(x),\ x = a,\ x = b$ је

$$P = \int_a^b |f(x) - g(x)| dx$$

На пример, (зад.91, стр.87)

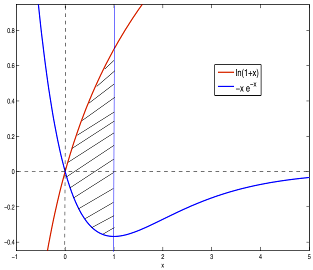
$y = \sin x,\ y = \cos x,\ 0 \le x \le \pi/4$

$$P = \int_0^{\pi/4} (cosx - \sin x) dx = \sin x + \cos x \Big|_0^{\pi/4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 0 - 1 = \sqrt{2} - 1$$

18 Збирка, зад.99, стр.88

$y = \ln(1+x), \;\; y = -xe^{-x}, \;\; x = 1$

Решење.



$$P = \int_0^1 \ln(1+x)dx + \int_0^1 xe^{-x}dx$$

$$P = (x+1)\ln(1+x) - x\Big|_0^1 + (-xe^{-x} - e^{-x})\Big|_0^1$$

$$P = 2\ln 2 - \frac{2}{e} = \ln 4 - \frac{2}{e}$$

19 Колоквијум, 2008

$$y = \frac{1}{\sqrt{x^2+2x+2}}, \quad y = 1, x = 0, \qquad P = ?$$

Решење.

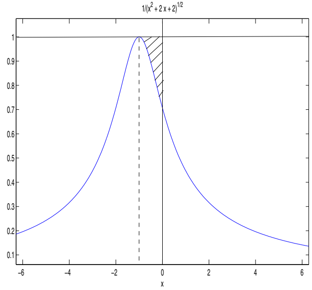
$$P = \int_{-1}^{0} \left(1 - \frac{1}{\sqrt{x^2+2x+2}}\right) dx$$

$$P = x \Big|_{-1}^{0} - \int_{-1}^{0} \frac{dx}{\sqrt{x^2+2x+1}}$$

$$P = 1 - \int_{-1}^{0} \frac{dx}{\sqrt{(x+1)^2+1}}$$

$$P = 1 - \ln(x + 1 + \sqrt{x^2+2x+2}) \Big|_{-1}^{0}$$

$$P = 1 - \ln(1+\sqrt{2}) \approx 0.1186$$



$$\sqrt{y}+\sqrt{x}=a \ \ (a>0), \ \ x=0, \ \ y=0$$

Решење.

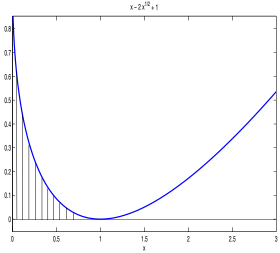
$$y=0, \qquad x=a^2$$

$$x=0, \qquad y=a^2$$

$$y=(a-\sqrt{x})^2$$

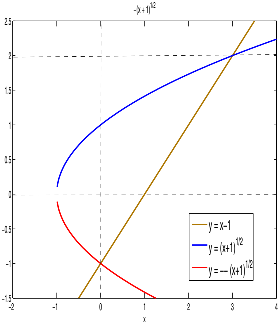
$$P=\int_0^{a^2}(a^2+x-2a\sqrt{x})dx$$

$$P=\frac{a^4}{6}$$



$y = x - 1, \;\; y^2 = x + 1$

Решење.



$$P_1 = \int_{-}^{0} 2\sqrt{x+1}dx$$

$$P_1 = \frac{4}{3}(x+1)^{3/2} \Big|_{-1}^{0}$$

$$P_2 = \int_{0}^{3} (\sqrt{x+1} - x + 1)dx$$

$$P_2 = \frac{2}{3}(x+1)^{3/2} - \frac{(x-1)^2}{2} \Big|_{0}^{3}$$

$$P = P_1 + P_2 = \frac{9}{2}$$

Друго решење.

$$P = \int_{-1}^{2} (y+1-y^2+1)dy = -\frac{y^3}{3} + \frac{y^2}{2} + 2y \Big|_{-1}^{2}$$

M2 - ВЕЖБЕ - 12. термин

Драган Ђорић

Факултет организационих наука

2009/2010

Садржај

1. Примена одређеног интеграла
   (дужина лука, запремина, површина)
2. Несвојствени интеграл

Дужина лука криве линије

$y = f(x),\ \ x \in [a, b] \qquad l = \int_a^b \sqrt{1 + f'^2(x)}\, dx$

$x = g(y),\ \ y \in [c, d] \qquad l = \int_c^d \sqrt{1 + g'^2(y)}\, dy$

$x = \varphi(t),\ \ y = \psi(t),\ \ t \in [t_1, t_2] \qquad l = \int_{t_1}^{t_2} \sqrt{\varphi'^2(t) + \psi'^2(t)}\, dt$

$\rho = \rho(\varphi),\ \ \varphi \in [\alpha, \beta] \qquad l = \int_\alpha^\beta \sqrt{\rho^2 + \rho'^2}\, d\varphi$

1 Збирка, зад.112, стр.89

$$f(x) = \frac{x^2}{2} - \frac{\ln x}{4}, \quad x \in [1,3]$$

Решење.

$$f'(x) = x - \frac{1}{4x}, \qquad f'^2(x) = x^2 + \frac{1}{16x^2} - \frac{1}{2}$$

$$1 + f'^2(x) = \frac{1}{2} + x^2 + \frac{1}{16x^2} = \left(x + \frac{1}{4x}\right)^2$$

$$\sqrt{1 + f'^2(x)} = x + \frac{1}{4x}$$

$$l = \int_1^3 \left(x + \frac{1}{4x}\right)\, dx = \frac{x^2}{2} + \frac{1}{4}\ln x \Big|_1^3$$

$$l = \frac{9}{2} + \frac{1}{4}\ln 3 - \frac{1}{2} = 4 + \frac{1}{4}\ln 3$$

2 Збирка, зад.115, стр.89

$y = \ln(x^2 - 1), \qquad 2 \le x \le 5$

Решење.

$$f'(x) = \frac{2x}{x^2-1}, \qquad 1 + f'^2(x) = 1 + \frac{4x^2}{(x^2-1)^2}$$

$$1 + f'^2(x) = \frac{(x^2-1)^2 + 4x^2}{(x^2-1)^2} = \frac{(x^2+1)^2}{(x^2-1)^2}$$

$$l = \int_2^5 \frac{x^2+1}{x^2-1} dx = \int_2^5 dx + 2\int_2^5 \frac{dx}{x^2-1} = x\Big|_2^5 + 2 \cdot \frac{1}{2} \ln\left|\frac{x-1}{x+1}\right| \Big|_2^5$$

$$l = 3 + \ln\frac{2}{3} - \ln\frac{1}{3} = 3 + \ln 2$$

3 Збирка, зад.117, стр.89

$$y = \ln(\sin x), \qquad \frac{\pi}{3} \le x \le \frac{2\pi}{3}$$

Решење.

$$y' = \cot x, \qquad 1 + y'^2(x) = 1 + \cot^2 x = \frac{1}{\sin^2 x}$$

$$l = \int_{\pi/3}^{2\pi/3} \frac{dx}{\sin x} = \frac{1}{2} \int_{\pi/3}^{2\pi/3} \frac{dx}{\sin \frac{x}{2} \cos \frac{x}{2}}$$

$$l = \int_{\pi/3}^{2\pi/3} \frac{d\left(\tan \frac{x}{2}\right)}{\tan \frac{x}{2}} = \ln\left(\tan \frac{x}{2}\right) \Big|_{\pi/3}^{2\pi/3}$$

$$l = \ln\sqrt{3} - \ln\frac{1}{\sqrt{3}} = \ln\sqrt{3} + \ln\sqrt{3} = \ln 3$$

4

$$y = \ln \frac{e^x - 1}{e^x + 1}, \qquad x \in [1, 2]$$

Решење.

$$y' - \frac{e^{2x}}{e^{2x} - 1}, \qquad 1 + y'^2 = \frac{(e^{2x} + 1)^2}{(e^{2x} - 1)^2}$$

$$l = \int_1^2 \frac{e^{2x} + 1}{e^{2x} - 1} dx = \int_e^{e^2} \frac{t^2 + 1}{t(t^2 - 1)} dt$$

$$l = \int_e^{e^2} \left(\frac{1}{t - 1} + \frac{1}{t + 1} - \frac{1}{t} \right) dt = \ln \frac{|t^2 - 1|}{|t|} \Big|_e^{e^2}$$

$$l = -1 + \ln(e^2 + 1)$$

5

$y = \dfrac{x^2}{2} - 1$ испод x-осе

Решење.

$y' = x$

$$l = \int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{1+x^2}\, dx = 2\int_0^{\sqrt{2}} \sqrt{1+x^2}\, dx = 2\int_0^{\sqrt{2}} \frac{1+x^2}{\sqrt{1+x^2}}\, dx$$

$$l = 2\int_0^{\sqrt{2}} \left(\frac{1}{\sqrt{1+x^2}} + \frac{x^2}{\sqrt{1+x^2}} \right) dx = 2\ln(x+\sqrt{1+x^2})\Big|_0^{\sqrt{2}} + 2J$$

За J: $u = x,\ dv = \dfrac{xdx}{\sqrt{1+x^2}},\ v = \sqrt{1+x^2}$

$$2J = 2x\sqrt{1+x^2}\Big|_0^{\sqrt{2}} - 2\int_0^{\sqrt{2}} \sqrt{1+x^2}\, dx = 2\sqrt{6} - l$$

$$l = 2\ln(\sqrt{2}+\sqrt{3}) + 2\sqrt{6} - l$$

$$l = \ln(\sqrt{2}+\sqrt{3}) + \sqrt{6}$$

6 Колоквијум, 2003, а и 2006

$$y = \sqrt{x^2 - 48} + 4\sqrt{6}\ln(x + \sqrt{x^2 - 48}), \qquad 7 \le x \le 8$$

Решење.

$$y' = \frac{x + 4\sqrt{6}}{\sqrt{x^2 - 48}}, \qquad \sqrt{1 + y'^2} = \sqrt{2} \cdot \frac{x + 2\sqrt{6}}{\sqrt{x^2 - 48}}$$

$$l = \int_7^8 \frac{\sqrt{2}}{2} \cdot \frac{d(x^2 - 48)}{\sqrt{x^2 - 48}} + \sqrt{2} \cdot 2\sqrt{6} \int_7^8 \frac{dx}{\sqrt{x^2 - (4\sqrt{3})^2}}$$

$$l = \frac{\sqrt{2}}{2} \cdot 2\sqrt{x^2 - 48}\,\Big|_7^8 + 2\sqrt{12}\ln(x + \sqrt{x^2 - 48})\,\Big|_7^8$$

$$l = 3\sqrt{2} + 4\sqrt{3}\ln\frac{3}{2}$$

7 Збирка, зад.123, стр.89

$x(t) = a\cos^3 t,\;\; y(t) = a\sin^3 t$ (Астроида)

Решење.

$$x'^2 = 9a^2\cos^4 t\sin^2 t,\;\; y'^2 = 9a^2\sin^4 t\cos^2 t$$

$$x'^2 + y'^2 = 9a^2\cos^2 t\sin^2 t(\cos^2 t + \sin^2 t)$$

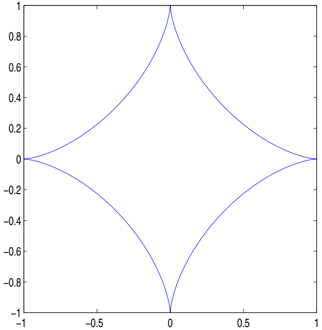
$$x'^2 + y'^2 = 9a^2\cos^2 t\sin^2 t$$

$$l = 4\cdot 3a\int_0^{\pi/2}\cos t\sin t\,dt = 12a\int_0^{\pi/2}\sin t d(\sin t)$$

$$l = 6a\;\sin^2 t\Big|_0^{\pi/2} = 6a$$

Зашто не може:

$$l = 3a\int_0^{2\pi}\cos t\sin t\,dt = \frac{3a}{2}\sin^2 t\Big|_0^{2\pi} = 0?$$



8 Збирка, зад.124, стр.89

$x = a(t - \sin t),\ \ y = a(1 - \cos t),\ \ t \in [0, 2\pi]$
(Циклоида)

Решење.

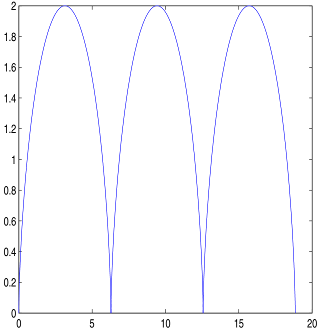
$$x' = a(1 - \cos t), \qquad y' = a \sin t$$

$$x'^2 + y'^2 = a^2(1 + \cos^2 t - 2\cos + \sin^2 t)$$

$$x'^2 + y'^2 = 2a^2(1 - \cos t) = 4a^2 \sin^2 \frac{t}{2}$$

$$l = \int_0^{2\pi} 2a \sin \frac{t}{2}\, dt = 2a \left(- \cos \frac{t}{2} \right) \cdot 2 \Big|_0^{2\pi}$$

$$l = 4a(1 + 1) = 8a$$



9 Колоквијум, 2002

$$x(t)=2\sqrt{2}\sqrt{1-t^2},\;\; y(t)=t\sqrt{1-t^2},\qquad t\in[0,1]$$

Решење.

$$x'(t)=-2\sqrt{2}\frac{t}{\sqrt{1-t^2}},\qquad y'(t)=\sqrt{1-t^2}-\frac{t^2}{\sqrt{1-t^2}}$$

$$x'^2+y'^2=\frac{(2t^2+1)^2}{1-t^2},\qquad \sqrt{x'^2+y'^2}=\frac{2t^2+1}{\sqrt{1-t^2}}$$

$$l=\int_0^1\frac{2t^2+1}{\sqrt{1-t^2}}dt=-2I+3\arcsin t\,\Big|_0^1=-2I+\frac{3}{2}\pi$$

$$I=\int_0^1\sqrt{1-t^2}dt=\int_0^{\pi/2}\cos^2 u du$$

$$I=\int_0^{\pi/2}\frac{1+\cos 2u}{2}du=\frac{1}{2}u\,\Big|_0^{\pi/2}+\frac{1}{4}\sin 2u\,\Big|_0^{\pi/2}=\frac{\pi}{4}$$

$$l=-2\cdot\frac{\pi}{4}+\frac{3}{2}\pi=\pi$$

Запремина ротационог тела

$$y = f(x),\ \ x \in [a, b] \qquad V_x = \pi \int_a^b f^2(x)dx,\ \ V_y = 2\pi \int_a^b |xf(x)|dx$$

$$x = g(y),\ \ y \in [c, d] \qquad V_y = \pi \int_c^d g^2(y)dy,\ \ V_x = 2\pi \int_c^d |yg(y)|dy$$

$$x = \varphi(t),\ \ y = \psi(t),\ \ t \in [\alpha, \beta] \qquad V_x = \pi \int_\alpha^\beta \psi^2(t)\varphi'(t)dt$$

$$\rho = \rho(\theta),\ \ \theta \in [\alpha, \beta] \qquad V = \frac{2\pi}{3} \int_\alpha^\beta \rho^3(\theta)\sin\theta d\theta$$

10

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad V_x = ?, \; V_y = ?$$

Решење.

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

$$V_x = \pi \int_{-a}^{a} y^2 dx = \frac{\pi b^2}{a^2} \int_{-a}^{a} (a^2 - x^2) dx = \frac{\pi b^2}{a^2} \left(a^2 x - \frac{x^3}{3} \right) \Big|_{-a}^{a} = \frac{4}{3} \pi a b^2$$

$$x^2 = \frac{a^2}{b^2}(b^2 - y^2)$$

$$V_y = \pi \int_{-b}^{b} x^2 dy = \frac{\pi a^2}{b^2} \int_{-b}^{b} (b^2 - y^2) dy = \frac{\pi a^2}{b^2} \left(b^2 y - \frac{y^3}{3} \right) \Big|_{-b}^{b} = \frac{4}{3} \pi a^2 b$$

За $a = b = R \qquad V_L = \frac{4}{3} R^3$

11

$y = \sin x, \quad x = 0, \quad y = 1 \qquad V_y = ?$

Решење.

$$V_y = \pi \int_0^1 \arcsin^2 y dy = \pi I$$

За I : $\quad u = \arcsin^2 y, \quad dv = dy, \quad du = 2 \arcsin y \cdot \dfrac{dy}{\sqrt{1-y^2}}$

$$I = y \arcsin y \Big|_0^1 - 2 \int_0^1 \arcsin y \cdot \frac{y dy}{\sqrt{1-y^2}} = \frac{\pi^2}{4} - 2J$$

За J : $\quad u = \arcsin y, \quad dv = \dfrac{y dy}{\sqrt{1-y^2}}, du = \dfrac{dy}{\sqrt{1-y^2}}, \quad v = -\sqrt{1-y^2}$

$$J = -\sqrt{1-y^2} \arcsin y \Big|_0^1 + \int_0^1 \sqrt{1-y^2} \cdot \frac{dy}{\sqrt{1-y^2}} = 1$$

$$I = \frac{\pi^2}{4} - 2, \qquad V_y = \frac{\pi}{4}(\pi^2 - 8)$$

Друго решење.

$\arcsin y = t, \;\; y = \sin t, \;\; dy = \cos t dt$

$$I = \int_0^{\pi/2} t^2 \cos t dt$$

Треће решење.

$$V_y = \left(\frac{\pi}{2}\right)^2 \pi \cdot 1 - V_y^*$$

$$V_y^* = 2\pi \int_0^{\pi/2} x \cdot sinx dx = 2\pi$$

12

$y = e^x, \;\; x = 0, \;\; x = 2, \;\; y = 0, \qquad V_y?$

Решење.

$$V_y = 2^2 \pi e^2 - \pi \int_1^{e^2} \ln^2 y dy$$

$$V_y = 2\pi(e^2 + 1)$$

13 Збирка, зад.146, стр.90

$y^2 = 2x,\ \ y = 2,\ \ x = 0, \qquad V_x = ?$

Решење.

$V_x = V_1 - V_2, \qquad V_1 = r^2 \pi H = 2^2 \pi \cdot 2$

$V_2 = \pi \int_0^2 2x dx = \pi\ x^2 \Big|_0^2 = 4\pi$

$V_x = 8\pi - 4\pi = 4\pi$

14 Збирка, зад.147, стр.90

$y = x, \;\; y = \dfrac{1}{x}, \;\; x = 3, \qquad V_x = ?$

Решење.

$V_x = V_1 + V_2$

$V_1 = \pi \displaystyle\int_0^1 y^2 dx = \pi \int_0^1 x^2 dx = \frac{1}{3}\pi$

$V_2 = \pi \displaystyle\int_1^3 \frac{1}{x^2} dx = \frac{2}{3}\pi$

$V_x = \pi$

15 Збирка, зад.155, стр.91

$y = 2x - x^2, y = 0, \qquad V_y = ?$

Решење.

$2x - x^2 = y,\ \ x^2 - 2x + y = 0,\ \ x_{1/2} = 1 \pm \sqrt{1-y}$

$V_y = V_1 - V_2$

$$V_1 = \pi \int_0^1 (1 + \sqrt{1-y})^2 dy = \frac{17}{6}\pi$$

$$V_2 = \pi \int_0^1 (1 - \sqrt{1-y})^2 dy = \frac{1}{6}\pi$$

$$V_y = \frac{8}{3}\pi$$

Друго решење.

$$V_y = 2\pi \int_0^2 x(2x - x^2)dx = 2\pi \int_0^2 (2x^2 - x^3)dx = 2\pi \left(\frac{2}{3}x^3 - \frac{x^4}{4} \right) \Big|_0^2 = \frac{8}{3}\pi$$

16

$y = 2x - x^2, y = 0, \qquad V_x = ?$

Решење.

$$V_x = \pi \int_0^2 (2x - x^2)^2 dx$$

$$V_x = \pi \int_0^2 (4x^2 - 4x^3 + x^4) dx$$

$$V_x = \frac{16}{15}\pi$$

17

$x^2 + y^2 = r^2, \;\; (x-r)^2 + y^2 = r^2, \qquad V = ?$ (заједничи део)

Решење.

$$V = 2\pi \int_0^{r/2} (r^2 - (x-r)^2) dx$$

$$V = \frac{5}{12}\pi^3$$

18

$y = x^2, \;\; 8x = y^2, \qquad V_y = ?$

Решење.

$$V_y = \pi \int_0^4 \left(y - \frac{y^4}{64} \right) dy$$

$$V_y = \frac{24}{5}\pi$$

Површина омотача рот. тела

$y = f(x),\ \ x \in [a,b] \qquad P_x = 2\pi \int_a^b y\sqrt{1+y'^2}\, dx$

$x = g(y),\ \ y \in [c,d] \qquad P_y = 2\pi \int_c^d x\sqrt{1+x'^2}\, dy$

$x = \varphi(t),\ \ y = \psi(t),\ \ t \in [\alpha,\beta] \qquad P_x = 2\pi \int_\alpha^\beta \psi(t)\sqrt{\varphi'^2(t)+\psi'^2(t)}\, dt$

$\rho = \rho(\theta),\ \ \theta \in [\alpha,\beta] \qquad P = 2\pi \int_\alpha^\beta \rho(\theta)\sqrt{\rho^2(\theta)+\rho'^2(\theta)}\sin\theta d\theta$

19 Збирка, зад.160, стр.?

$y = x^3,\ 0 \leq x \leq 1,$ око x-осе

Решење.

$y' = 3x^2, \qquad y'^2 = 9x^4$

$$P_x = 2\pi \int_0^1 x^3 \sqrt{1 + 9x^4}\, dx$$

$$P_x = \frac{2\pi}{36} \int_0^1 (1 + 9x^4)^{1/2} d(1 + 9x^4)$$

$$P_x = \frac{\pi}{18} \cdot \frac{2}{3} (1 + 9x^4)^{3/2} \Big|_0^1$$

$$P_x = \frac{\pi}{27} \left(10^{3/2} - 1\right)$$

20 Колоквијум, 2009

$$y = x^2 - \frac{\ln x}{8}, \quad \sqrt{e} \le x \le e, \qquad \text{око } x \text{ -осе}$$

Решење.

$$y' = 2x - \frac{1}{8x}, \qquad 1 + y'^2 = 1 = 4x^2 + \frac{1}{64x^2} - \frac{1}{2} = \left(2x + \frac{1}{8x}\right)^2$$

$$P_x = 2\pi \int_{\sqrt{e}}^{e} \left(x^2 - \frac{\ln x}{8}\right)\left(2x + \frac{1}{8x}\right) dx$$

$$P_x = 2\pi \int_{\sqrt{e}}^{e} \left(2x^3 + \frac{x}{8} - \frac{1}{4} x \ln x - \frac{1}{64} \frac{\ln x}{x}\right) dx$$

$$\int x \ln x dx = \frac{x^2}{2} \ln x - \frac{x^2}{4}$$

$$P_x = ??$$

21

$y = x^2, \;\; x = y^2, \qquad V_x = ?, \;\; P_x = ?$

Решење.

$$V_x = V_1 - V_2 = \int_0^1 \pi(x - x^4)dx = \frac{3}{10}\pi$$

$$P_x = P_1 + P_2$$

За $P_1: \qquad x = y^2, \;\; y = \sqrt{x}, \;\; y' = \frac{1}{2\sqrt{x}}$

$$P_1 = 2\pi \int_0^1 \sqrt{x}\sqrt{1 + \frac{1}{4x}}\, dx = \pi \int_0^1 \sqrt{4x+1}dx$$

$$P_1 = \frac{\pi}{6}(4x+1)^{3/2}\Big|_0^1 = \frac{\pi}{6}(5\sqrt{5} - 1)$$

За $P_2: \qquad y = x^2, \;\; y' = 2x$

$$P_2 = 2\pi \int_0^1 x^2\sqrt{1+4x^2}\, dx = \cdots = \frac{9\sqrt{5}}{16}\pi - \frac{1}{32}\pi\ln(2+\sqrt{5})$$

Додатак - $I = \int_0^1 x^2\sqrt{1+4x^2}\,dx$

$$u = x,\ \ dv = x\sqrt{1+4x^2}\,dx, \qquad du = dx,\ \ v = \frac{1}{12}(1+4x^2)^{3/2}$$

$$I = \frac{x}{12}(1+4x^2)^{3/2}\Big|_0^1 - \frac{1}{12}\int_0^1 (1+4x^2)^{3/2}dx$$

$$I = \frac{5}{12}\sqrt{5} - \frac{1}{12}\int_0^1 (1+4x^2)\sqrt{1+4x^2}\,dx = \frac{5}{12}\sqrt{5} - \frac{1}{12}\int_0^1 \sqrt{1+4x^2}\,dx - \frac{4}{12}I$$

$$I = \frac{5}{16}\sqrt{5} - \frac{1}{16}J, \qquad J = \int_0^1 \sqrt{1+4x^2}\,dx$$

$$u = \sqrt{1+4x^2},\ \ dv = dx, \qquad du = \frac{4x\,dx}{\sqrt{1+4x^2}},\ \ v = x$$

$$J = x\sqrt{1+4x^2}\,\Big|_0^1 - 4\int_0^1 \frac{x^2dx}{\sqrt{1+4x^2}} = \sqrt{5} - J + \frac{1}{2}\ln(2x+\sqrt{1+4x^2})$$

$$J = \frac{\sqrt{5}}{2} + \frac{1}{4}\ln(2+\sqrt{5}), \qquad I = \frac{9}{16}\sqrt{5} - \frac{1}{64}\ln(2+\sqrt{5})$$

22

$$y = \sin x, \qquad x \in [0, \pi]$$

Решење.

$$y' = \sin x, \qquad 1 + f'^2 = 1 + \cos^2 x$$

$$P = 2\pi \int_0^\pi \sin x \sqrt{1 + \cos^2 x}\, dx = -2\pi \int_0^\pi \sqrt{1 + \cos^2 x} d(\cos x)$$

$$\cos x = t, \qquad P = -2\pi \int_{-1}^1 \sqrt{1 + t^2}\, dt = 2\pi \int_0^1 \sqrt{1 + t^2}\, dt$$

$$P = 4\pi \int_0^1 \sqrt{1 + t^2}\, dt = 4\pi I, \qquad I = \int_0^1 \sqrt{1 + t^2}\, dt$$

$$I = \int_0^1 \frac{1 + t^2}{\sqrt{1 + t^2}} dt = \int_0^1 \frac{dt}{\sqrt{1 + t^2}} + J, \text{ За } J: \ u = t, \ dv = \frac{tdt}{\sqrt{1 + t^2}}$$

$$J = \int_0^1 \frac{t^2 dt}{\sqrt{1 + t^2}} = t\sqrt{1 + t^2}\Big|_0^1 - I, \qquad 2I = \ln(1 + \sqrt{2}) + \sqrt{2}$$

$$P = 4\pi I = 2\pi \ln(1 + \sqrt{2}) + 2\sqrt{2}\pi = \pi(\ln(3 + 2\sqrt{2}) + 2\sqrt{2})$$

M2 - ВЕЖБЕ - 13. термин

Драган Ђорић

Факултет организационих наука

2009/2010

Двојни интеграли

1. Декартове координате
2. Смене променљивих
3. Поларне координате
4. Запремине тела
5. Површине равних фигура
6. Површина дела површи

1. Декартове координате

$\mathcal{D} = [a,b] \times [c,d]$

$$\iint_{\mathcal{D}} f(x,y)dxdy = \int_a^b dx \int_c^d f(x,y)dy = \int_c^d dy \int_a^b f(x,y)dx$$

$\mathcal{D} = \{(x,y) \mid x \in [a,b],\ y_1(x) \leq y \leq y_2(x)\}$

$$\iint_{\mathcal{D}} f(x,y)dxdy = \int_a^b dx \int_{y_1(x)}^{y_2(x)} f(x,y)dy$$

$\mathcal{D} = \{(x,y) \mid y \in [c,d],\ x_1(y) \leq x \leq x_2(y)\}$

$$\iint_{\mathcal{D}} f(x,y)dxdy = \int_c^d dy \int_{x_1(y)}^{x_2(y)} f(x,y)dx$$

1

$$\iint_{\mathcal{D}} \frac{x^2}{1+y^2} dxdy, \qquad \mathcal{D} = [0,1] \times [0,1]$$

Решење.

$$I = \int_0^1 x^2 dx \cdot \int_0^1 \frac{dy}{1+y^2}$$

$$I = \frac{x^3}{3} \Big|_0^1 \cdot \arctan y \Big|_0^1 dy$$

$$I = 1 \cdot \frac{\pi}{4} = \frac{\pi}{4}$$

2

$$\iint_{\mathcal{D}} \frac{dxdy}{(x+y+1)^2}, \qquad \mathcal{D} = [0,1] \times [0,1]$$

Решење.

$$I = \int_0^1 \left[\int_0^1 \frac{dy}{(x+y+1)^2} \right] dx$$

$$I = \int_0^1 \left[\frac{-1}{x+y+1} \Big|_0^1 \right] dx$$

$$I = \int_0^1 \left[\frac{-1}{x+2} + \frac{1}{x+1} \right] dx$$

$$I = -\ln(x+2) + \ln(x+1) \Big|_0^1 = \ln \frac{x+1}{x+2} \Big|_0^1$$

$$I = \ln \frac{4}{3}$$

$$\iint_{\mathcal{D}} (x+y)dxdy, \qquad \mathcal{D}: \;\; y^2 = 2x, \;\; x^2 = 2y$$

Решење.

$$I = \int_0^2 \left[\int_{x^2/2}^{\sqrt{2x}} (x+y)dy \right] dx$$

$$I = \int_0^2 x \cdot \left(y \,\Big|_{x^2/2}^{\sqrt{2x}} \right) \, dx + \int_0^2 \left(\frac{y^2}{2} \,\Big|_{x^2/2}^{\sqrt{2x}} \right) \, dx$$

$$I = \int_0^2 \left(x\sqrt{2x} - \frac{1}{2}x^3 \right) \, dx + \int_0^2 \left(x - \frac{1}{4}x^4 \right) dx$$

$$I = \frac{8}{5}$$

4

$$\iint_{\mathcal{D}} (x^2+y^2)dxdy, \qquad \mathcal{D}: \ y=x, \ y=x+1, \ y=1, \ y=3$$

Решење.

Најпре по x, па по y!

$$I = \int_1^3 \left[\int_{y-1}^{y} (x^2+y^2)dx\right] dy = \int_1^3 I(y)dy$$

$$I(y) = \int_{y-1}^{y} (x^2+y^2)dx = \frac{x^3}{3}\Big|_{y-1}^{y} + y^2\cdot x\Big|_{y-1}^{y}$$

$$I(y) = \frac{y^3}{3} - \frac{1}{3}(y-1)^3 + y^2(y-y+1) = 2y^2 - y + \frac{1}{3}$$

$$I = \int_1^3 \left(2y^2 - y + \frac{1}{3}\right) dy$$

$$I = 14$$

5

$$\iint_{\mathcal{D}} xydxdy, \qquad \mathcal{D}: \; x^2+y^2=1, \; x^2+y^2=2x, \; y=0$$

Решење.

$$I = \int_0^{\sqrt{3}/2} y \cdot I(y)dy, \qquad I(y) = \int_{1-\sqrt{1-y^2}}^{\sqrt{1-y^2}} xdx$$

$$I(y) = \frac{x^2}{2}\Big|_{1-\sqrt{1-y^2}}^{\sqrt{1-y^2}}$$

$$I(y) = \frac{1}{2}(1-y^2) - \frac{1}{2}(1+1-y^2-2\sqrt{1-y^2}) = -\frac{1}{2} + \sqrt{1-y^2}$$

$$I = -\frac{1}{2}\int_0^{\sqrt{3}/2}(y - 2y\sqrt{1-y^2})dy$$

$$I = \frac{5}{48}$$

6 Збирка, зад.25, стр.99

$$\iint_{\mathcal{D}} x^2y^2\sqrt{1-x^3-y^3}\,dxdy, \qquad \mathcal{D}=\{(x,y)\ :\ x\geq 0, y\geq 0, x^3+y^3\leq 1\}$$

Решење.

$$I=\int_0^1 x^2\cdot I(x)dx, \qquad I(x)=\int_0^{\sqrt[3]{1-x^3}} y^2\sqrt{1-x^3-y^3}dy$$

$$I(x)=\frac{1}{3}\cdot\frac{2}{3}(1-x^3-y^3)^{3/2}\Big|_0^{\sqrt[3]{1-x^3}}$$

$$I(x)=\frac{2}{9}(1-x^3)^{3/2}$$

$$I=\frac{2}{9}\int_0^1 x^2(1-x^3)^{3/2}dx=\frac{2}{9}\cdot\frac{1}{3}\int_0^1(1-x^3)^{3/2}d(1-x^3)$$

$$I=\frac{4}{135}$$

7 Колоквијум, 2002

$$I = \iint_{\mathcal{D}} x \cdot \sin|y - x^2| dxdy, \qquad \mathcal{D} = \{(x,y) \ : \ 0 \leq x \leq \sqrt{\pi/2}, \ 0 \leq y \leq \pi\}$$

Решење.

$$|y - x^2| = \begin{cases} y - x^2, & y \geq x^2 \\ x^2 - y, & y < x^2 \end{cases} = \begin{cases} y - x^2, & (x,y) \in \mathcal{D}_1 \\ x^2 - y, & (x,y) \in \mathcal{D}_2 \end{cases}$$

$$I = \iint_{\mathcal{D}_1} f(x,y) dxdy + \iint_{\mathcal{D}_2} f(x,y) dxdy = I_1 + I_2$$

$$I_1 = \int_0^{\sqrt{\pi/2}} x dx \int_{x^2}^{\pi} \sin(y - x^2) dy = \int_0^{\sqrt{\pi/2}} x(\cos x^2 + 1) dx$$

$$I_1 = \frac{1}{2}\sin x^2 + \frac{x^2}{2} \Big|_0^{\sqrt{\pi/2}} = \frac{1}{2} + \frac{\pi}{4}$$

$$I_2 = \int_0^{\sqrt{\pi/2}} x dx \int_{x^2}^{\pi} \sin(x^2 - y) dy = \int_0^{\sqrt{\pi/2}} x(1 - \cos x^2) dx$$

$$I_2 = \frac{x^2}{2} - \frac{1}{2}\sin x^2 \Big|_0^{\sqrt{\pi/2}} = \frac{\pi}{4} - \frac{1}{2}$$

$$I = I_1 + I_2 = \frac{1}{2} + \frac{\pi}{4} + \frac{\pi}{4} - \frac{1}{2} = \frac{\pi}{2}$$

2. Смене промењивих

$$F : \mathcal{G} \to \mathcal{D}, \qquad F : (u, v) \mapsto (x, y)$$

$$x = x(u, v),\ \ y = y(u, v), \qquad dxdy = |J|dudv,\ \ J = \frac{D(x, y)}{D(u, v)} = \begin{vmatrix} x'_u & x'_v \\ y'_u & y'_v \end{vmatrix}$$

$$\iint_{\mathcal{D}} f(x, y)dxdy = \iint_{\mathcal{G}} f(x(u, v), y(u, v))|J|dudv$$

Важи

$$J = \frac{D(x, y)}{D(u, v)} = \frac{1}{\dfrac{D(u, v)}{D(x, y)}}$$

$$\iint_{\mathcal{D}} (x^2 - y^2)dxdy, \qquad \mathcal{D}\,;\; x+y=1,\;\; x+y=3,\;\; 2x-y=2,\;\; 2x-y=-1$$

Решење.

$$x+y=u,\;\; x-y=v, \qquad x=\frac{u+v}{2},\;\; y=\frac{u-v}{2}$$

$$x+y=1 \to u=1, \qquad x+y=3 \to u=3,$$

$$2x-y=2 \to u+3v=4, \qquad 2x-y=-1 \to u+3v=-2$$

$$\mathcal{G}:\;\; u=1,\;\; u=3,\;\; u=3v=-2,\;\; u+3v=4$$

$$J=\begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix}$$

$$I=\int_1^3 du \int_{-(u+2)/3}^{(4-u)/3} uv|J|dv = \frac{1}{2}\int_1^3 u\left[\frac{v^2}{2}\,\Big|_{-(u+2)/3}^{(4-u)/3}\right]du$$

$$I=-\frac{14}{9}$$

Друго решење.

$$x + y = u,\;\; 2x - y = v, \qquad x = \frac{u+v}{3},\;\; y = \frac{2u-v}{3},\;\; J = -\frac{1}{3}$$

$$u \in [1,3],\;\; v \in [-1,2]$$

$$I = \int_1^3 du \int_{-1}^2 \frac{u}{3}(2v-u) \cdot \frac{1}{3} dv$$

$$I = -\frac{14}{9}$$

9

$$\iint_{\mathcal{D}} \cos\frac{x-y}{x+y}\,dxdy, \qquad \mathcal{D}:\;\; x+y=1,\;\; x=0,\;\; y=0$$

Решење.

$$u=x-y,\;\; v=x+y, \qquad x=\frac{u+v}{2},\;\; y=\frac{v-u}{2},\;\; J=\frac{1}{2}$$

$$\mathcal{G}:\;\; v=1,\;\; u=v,\;\; u=-v$$

$$I=\frac{1}{2}\int_0^1 dv\int_{-v}^{v}\cos\frac{u}{v}du$$

$$I=\frac{1}{2}\int_0^1\left(v\sin\frac{u}{v}\right)\Big|_{-v}^{v}\,dv$$

$$I=\sin 1\cdot\int_0^1 vdv=\frac{1}{2}\sin 1$$

10

$$\iint_{\mathcal{D}} \sin\frac{\pi xy}{a^2}\,dxdy,\ \ \mathcal{D}\ :\ y=\alpha x,\ y=\beta x,\ xy=a,\ x>0, a>0, \beta>\alpha>0$$

Решење.

$$\frac{y}{x}=u,\ \ xy=v,\qquad x=\sqrt{\frac{v}{u}},\ \ y=\sqrt{uv},\qquad (u,v)\in[\alpha,\beta]\times[0,a]$$

$$\frac{D(u,v)}{D(x,y)}=\begin{vmatrix}-y/x^2 & 1/x\\ y & x\end{vmatrix}=-\frac{y}{x},\qquad \frac{D(x,y)}{D(u,v)}=-\frac{1}{2}\cdot\frac{x}{y}=-\frac{1}{2u}$$

$$I=\frac{1}{2}\int_\alpha^\beta\frac{du}{u}\int_0^a\sin\frac{\pi v}{a}dv$$

$$I=\frac{1}{2}\ln\frac{\beta}{\alpha}\left[-\frac{a}{\pi}\cos\frac{\pi v}{a}\Big|_0^a\right]=\frac{a}{\pi}\ln\frac{\beta}{\alpha}$$

3. Поларне координате

$x = \rho\cos\varphi,\ \ y = \rho\sin\varphi, \qquad J = \rho$

За $x^2 + y^2 \leq R^2$ je $(\rho, \varphi) \in [0, R] \times [0, 2\pi]$

Елиптичке координате

$x = a\rho\cos\theta,\ \ y = b\rho\sin\theta, \qquad dxdy = ab\rho d\rho d\theta, \qquad J = ab\rho$

11 Збирка, зад.36, стр.100

$$\iint_D \frac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}dxdy, \qquad D = \{(x,y) \mid \pi^2/9 \le x^2+y^2 \le \pi^2\}$$

Решење.

$$x = \rho\cos\varphi,\ \ y = \rho\sin\varphi, \qquad J = \rho$$

$$\mathcal{G} = [0, 2\pi] \times [\pi/3, \pi]$$

$$I = \int_0^{2\pi} d\varphi \int_{\pi/3}^{\pi} \sin\rho d\rho$$

$$I = 2\pi \cdot (-\cos\varphi)\Big|_{\pi/3}^{\pi}$$

$$I = 3\pi$$

12

$$I = \iint_{\mathcal{D}} (1 - 2x - 3y)\,dx\,dy, \qquad \mathcal{D}\ :\ x^2 + y^2 \le R$$

Решење.

$$I = \int_0^{2\pi} d\varphi \int_0^R (1 - 2\rho\cos\varphi - 3\rho\sin\varphi)\rho\,d\rho$$

$$I = \int_0^{2\pi} d\varphi \left(\frac{\rho^2}{2} - \frac{2}{3}\rho^3\cos\varphi - \rho^3\sin\varphi \right) \Big|_0^R$$

$$I = \frac{R^2}{6}(3\varphi - 4R\sin\varphi + 6R\cos\varphi) \Big|_0^{2\pi}$$

$$I = \pi R^2$$

13 Колоквијум, 2004

$$\iint_{\mathcal{D}} \frac{dxdy}{(x^2+y^2)^2}, \qquad \mathcal{D}\ :\ x^2+y^2=4x,\ \ x^2+y^2=8x,\ \ y=0,\ \ y=x$$

Решење.

$x=\rho\cos\varphi,\ \ y=\rho\sin\varphi, \qquad \rho\in[0,\pi/4],\ \ \varphi\in[4\cos\varphi, 8\cos\varphi]$

$$I=\iint_{\mathcal{G}} \frac{\rho d\rho d\varphi}{\rho^4}=\int_0^{\pi/4} d\varphi \int_{4\cos\varphi}^{8\cos\varphi} \frac{d\rho}{\rho^3}$$

$$I=-\frac{1}{2}\int_0^{\pi/4} d\varphi\cdot\frac{1}{\rho^2}\Big|_{4\cos\varphi}^{8\cos\varphi}=-\frac{1}{2}\int_0^{\pi/4}\left(\frac{1}{64\cos^2\varphi}-\frac{1}{16\cos^2\varphi}\right)d\varphi$$

$$I=-\frac{1}{2}\left(\frac{1}{64}-\frac{1}{16}\right)\int_0^{\pi/4}\frac{d\varphi}{\cos^2\varphi}$$

$$I=-\frac{1}{2}\cdot\frac{1-4}{64}\cdot\tan\varphi\Big|_0^{\pi/4}=\frac{3}{128}$$

14 Збирка, зад.37, стр.100

$$I = \iint_{\mathcal{D}} \sqrt{R^2 - x^2 - y^2}\, dxdy, \qquad \mathcal{D} \,:\, x^2 + y^2 \leq Rx,\;\; y \geq 0$$

Решење.

$\rho \in [0, R\cos\varphi],\;\; \varphi \in [0, \pi/2]$

$$I = \int_0^{\pi/2} d\varphi \int_0^{R\cos\varphi} \sqrt{R^2 - \rho^2}\, \rho d\rho = -\frac{1}{3}\int_0^{\pi/2} d\varphi \sqrt{(R^2 - \rho^2)^3}\,\Big|_0^{R\cos\varphi}$$

$$I = -\frac{1}{3}\int_0^{\pi/2} \left(\sqrt{(R^2 - R^2\cos^2\varphi)^3} - R^2\right)\, d\varphi$$

$$I = -\frac{R^2}{3}\int_0^{\pi/2} (\sin^2\varphi - 1)d\varphi$$

$$I = -\frac{R^2}{3}\left(-\frac{1}{3}\sin^2\varphi\cos\varphi - \frac{2}{3}\cos\varphi - \varphi\right)\,\Big|_0^{\pi/2}$$

$$I = \frac{R^3}{6}\left(\pi - \frac{4}{3}\right)$$

15 Збирка, зад.38, стр.100

$$I = \iint_{\mathcal{D}} \arctan\frac{y}{x}dxdy, \qquad \mathcal{D} : x^2+y^2=1,\ x^2+y^2=9,\ y=\frac{\sqrt{3}}{3}x,\ y=\sqrt{3}x$$

Решење.

$$\rho \in [1,3],\ \varphi \in [\pi/6, \pi/3], \qquad \frac{y}{x} = tan\varphi,\ \arctan\frac{y}{x} = \varphi$$

$$I = \int_{\pi/6}^{\pi/3} \varphi d\varphi \int_1^3 \rho d\rho$$

$$I = \frac{\varphi^2}{2}\Big|_{\pi/6}^{\pi/3} \cdot \frac{\rho^2}{2}\Big|_1^3$$

$$I = \frac{\pi^2}{6}$$

16

$$\int_{\mathcal{D}} \sqrt{4-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\, dxdy, \qquad \mathcal{D}\ :\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ \ \frac{x^2}{4a^2}+\frac{y^2}{4b^2}=1,\ \ x\geq 0,\ \ y\geq 0$$

Решење.

$x = a\rho\cos\theta,\ \ y = b\rho\sin\theta$

$\rho\in[1,2],\ \ \theta\in[0,\pi/2], \qquad J = ab\rho$

$$I = ab\int_0^{\pi/2} d\theta \int_1^2 \sqrt{4-\rho^2}\ \rho d\rho$$

$$I = -\frac{1}{2}\cdot ab\cdot\frac{\pi}{2}\int_1^2 \sqrt{4-\rho^2}d(4-\rho^2)$$

$$I = -ab\cdot\frac{\pi}{2}\cdot\frac{1}{3}\sqrt{(4-\rho^2)^3}\ \Big|_1^2$$

$$I = \frac{\sqrt{3}}{2}ab\pi$$

4. Запремине тела

Запремина цилиндричног тела чија је једна основа фигура $\mathcal{D}$ у равни Oxy, а друга основа површ $z = f(x, y)$

$$V = \iint_{\mathcal{D}} f(x, y)dxdy$$

17

$T: \quad z = x^2 + y^2, \;\; z = 2(x^2 + y^2), \;\; y = x, \;\; y^2 = x, \qquad V = ?$

Решење.

$$V = V_1 - V_2$$

$$V_1 = \iint_{\mathcal{D}} 2(x^2 + y^2)dxdy, \qquad V_2 = \iint_{\mathcal{D}} (x^2 + y^2)dxdy$$

$$\mathcal{D} = \{(x, y) \mid 0 \leq x \leq 1, \; x \leq y \leq \sqrt{x}\}$$

$$V = \iint_{\mathcal{D}} (x^2 + y^2)dxdy = \int_0^1 dx \int_x^{\sqrt{x}} (x^2 + y^2)dy$$

$$V = \frac{3}{35}$$

18 Колоквијум, 2003

$T: \;\; x^2+y^2=2x, \;\; z=2x+y, z=0, y\geq 0, \qquad V=?$

Решење.

$x=1+\rho\cos\varphi, \;\; y=\rho\sin\varphi$

$$V=\int_0^{\pi} d\varphi \int_0^1 (2+2\rho\cos\varphi+\rho\sin\varphi)\rho d\rho$$

$$V=\int_0^{\pi}\left(1+\frac{2}{3}\cos\varphi+\frac{1}{3}\sin\varphi\right)=\pi+\frac{2}{3}$$

Друго решење.

$x=\rho\cos\varphi, \;\; y=\rho\sin\varphi$

$$V=\int_0^{\pi/2} d\varphi \int_0^{2\cos\varphi} (2\cos\varphi+\sin\varphi)\rho^2 d\rho$$

$$V=\frac{16}{3}\int_0^{\pi/2}\cos^4\varphi d\varphi-\frac{8}{3}\int_0^{\pi/2}\cos^3\varphi d(\cos\varphi)=\frac{16}{3}\cdot\frac{3\pi}{16}-\frac{8}{3}\cdot\frac{-1}{4}=\pi+\frac{2}{3}$$

19 Колоквијум, ??

$$T: \quad z=0, \;\; z=\frac{x^2+y^2}{2}, \;\; (x-1)^2+y^2=1, \qquad V=?$$

Решење.

$x = 1+\rho\cos\varphi, \;\; y=\rho\sin\varphi, \;\; \rho\in[0,1], \;\; \varphi\in[0,2\pi]$

$$V=\iint_{\mathcal{D}}\frac{x^2+y^2}{2}dxdy=\int_0^{2\pi}d\varphi\int_0^1\frac{(1+\rho\cos\varphi)^2+(\rho\sin\varphi)^2}{2}\cdot\rho d\rho$$

$$V=\int_0^{2\pi}d\varphi\int_0^1\left(\frac{\rho}{2}+\rho^2\cos\varphi+\frac{\rho^3}{2}\right)d\rho$$

$$V=\int_0^{2\pi}d\varphi\left(\frac{\rho^2}{4}+\frac{\cos\varphi}{3}\rho^3+\frac{\rho^4}{8}\right)\Big|_0^1$$

$$V=\int_0^{2\pi}\left(\frac{3}{8}+\frac{\cos\varphi}{3}\right)d\varphi=\left(\frac{3}{8}\varphi+\frac{\sin\varphi}{3}\right)\Big|_0^{2\pi}=\frac{3}{4}\pi$$

Друго решење.

$x = \rho \cos \varphi, \;\; y = \rho \sin \varphi, \;\; \rho \in [0, 2\cos\varphi], \;\; \varphi \in [-\pi/2, \pi/2]$

$$V = \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2\cos\varphi} \frac{\rho^3}{2} d\varphi = 2\int_{-\pi/2}^{\pi/2} \cos^4 \varphi d\varphi$$

$$V = 2\int_{-\pi/2}^{\pi/2} \left(\frac{1+\cos 2\varphi}{2}\right)^2 d\varphi$$

$$V = 2\left(\frac{3}{8}\varphi + \frac{\sin 2\varphi}{4} + \frac{\sin 4\varphi}{32}\right) \Big|_{-\pi/2}^{\pi/2} = \frac{3}{4}\pi$$

5. Површине равних фигура

Површину $P(\mathcal{D})$ фигуре $\mathcal{D}$ можемо добити ако у двојном интегралу на области $\mathcal{D}$ узмемо функцију $f(x,y)=1$

$$P(\mathcal{D}) = \iint_{\mathcal{D}} dxdy$$

20

$\mathcal{D}: \quad x^2 + y^2 \leq 2x, \;\; |y| \leq x, \qquad P(\mathcal{D}) = ?$

Решење.

$x = \rho\cos\varphi, \;\; y = \rho\sin\varphi, \qquad J = \rho$

$\mathcal{G}: -\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{4}, \;\; 0 \leq \rho \leq 2\cos\varphi$

$$P(\mathcal{D}) = \iint_{\mathcal{D}} dxdy = \iint_{\mathcal{G}} \rho d\varphi d\rho$$

$$P(\mathcal{D}) = 2\int_0^{\pi/4} d\varphi \int_0^{2\cos\varphi} \rho d\rho = 2 \cdot \frac{\pi}{4} \cdot \frac{\rho^2}{2}\Big|_0^{2\cos\varphi}$$

$$P(\mathcal{D}) = 4\int_0^{\pi/4} \cos^2\varphi d\varphi = \frac{\pi}{2} + 1$$

21

$$\mathcal{D}: \ \left(\frac{x}{a}+\frac{y}{b}\right)^2 = \frac{x}{a}-\frac{y}{b}, \ \ y=0, \qquad P(\mathcal{D}) = ?$$

Решење.

$$P(\mathcal{D}) = \iint_{\mathcal{D}} dx dy$$

$$u = \frac{x}{a}+\frac{y}{b}, \qquad v = \frac{x}{a}-\frac{y}{b}$$

$$\mathcal{G}: \ \ v = u^2, \ \ v = u, \qquad J = -\frac{ab}{2}$$

$$P(\mathcal{D}) = \frac{ab}{2}\int_0^1 du \int_{u^2}^{u} dv = \frac{ab}{2}\int_0^1 (u-u^2) du$$

$$P(\mathcal{D}) = \frac{ab}{2}\cdot\frac{1}{6} = \frac{ab}{12}$$

6. Површина дела површи

Површина дела површи $z(x,y)$ над $\mathcal{D}$ је дата са

$$P = \iint_D \sqrt{1 + z_x'^2 + z_y'^2} dxdy$$

22

$z = x^2 + y^2, \quad \mathcal{D}: \; x^2 + y^2 \leq 1, \quad P = ?$

Решење.

$$z'_x = 2x, \quad z'_y = 2y, \quad 1 + z'^2_x + z'^2_y = 1 + 4x^2 + 4y^2$$

$$P = \iint_{\mathcal{D}} \sqrt{1 + 4x^2 + 4y^2}\, dxdy$$

$$x = \rho\cos\varphi, \;\; y = \rho\sin\varphi, \quad J = \rho, \;\; 1 + 4x^2 + 4y^2 = 1 + 4\rho^2$$

$$P = \int_0^{2\pi} d\varphi \int_0^1 \sqrt{1 + 4\rho^2}\rho d\rho$$

$$P = \frac{\pi}{6}(5^{3/2} - 1)$$

23

$$x^2+y^2+x^2=a^2, \qquad \mathcal{D}:\ \ x^2+y^2=ax$$

Решење.

$$z=\sqrt{a^2-x^2-y^2}, \qquad \mathcal{D}:\ \ (x-a/2)^2+y^2=a^2/4$$

$$z'_x=\frac{-x}{\sqrt{a^2-x^2-y^2}}, \qquad z'_y=\frac{-y}{\sqrt{a^2-x^2-y^2}}$$

$$P=2\iint_{\mathcal{D}}\sqrt{1+z_x'^2+z_y'^2}\,dxd=2\iint_{\mathcal{D}}\frac{adxdy}{\sqrt{a^2-x^2-y^2}}$$

$$x=\rho\cos\varphi,\ \ y=\rho\sin\varphi$$

$$\mathcal{G}:\ \ \rho=a\cos\varphi,\ \ \varphi\in[0,\pi/2]$$

$$P=2\int_0^{\pi/2}\int_0^{a\cos\varphi}\frac{a\rho d\rho}{\sqrt{a^2-\rho^2}}d\varphi$$

$$P=2a^2\int_0^{\pi/2}(1-\sin\varphi)d\varphi=(\pi-2)a^2$$