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**МАТЕМАТИКА 2**  
**Збирка задатака и примери колоквијума**

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**РЕШЕЊА ОДАБРАНИХ ЗАДАТАКА – ИНТЕГРАЛИ**

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Други део (144 задатка), 4.5.2011.

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# ЗАДАЦИ

## 1 Неодређени интеграл

$F : (a, b) \rightarrow \mathbb{R}$  је примитивна функција за  $f$  ако је

$$\forall x \in (a, b) \quad F'(x) = f(x)$$

$$\int f(x)dx = \{F \mid F \text{ је прим. функција за } f\} = \{F + C, C \in \mathbb{R}\} = F(x) + C$$

$F$	$\sin x$	$e^x$	$-e^{-x}$	$\arctan x$	$\ln x$	$\frac{1}{n+1}x^{n+1}$	$\dots$
$f$	$\cos x$	$e^x$	$e^{-x}$	$\frac{1}{1+x^2}$	$\frac{1}{x}$	$x^n$	$\dots$

Свака непрекидна функција има примитивну

Примитивна функција елементарне функције не мора бити елементарна функција. На пример, за функције

$$x \mapsto \frac{\sin x}{x}, \quad x \mapsto \frac{e^x}{x}, \quad x \mapsto e^{-x^2}, \quad x \mapsto \sin x^2$$

примитивне функције нису елементарне

### 1.1. [Зад.10, стр.73]

За функцију  $f : x \mapsto \frac{1}{1+x^2}$  одредити примитивну  $F$  за коју је  $F(1) = \pi$ .

### 1.2. [Зад.4, стр.73]

Одредити бар једну примитивну функцију за функцију  $f : x \mapsto e^{|x|}$

### 1.3. [Није из Збирке]

Доказати да су  $F : x \mapsto \ln(x + \sqrt{x^2 + a^2})$  ( $a \neq 0$ ) и  $G : x \mapsto \ln(x + \sqrt{x^2 - a^2})$  ( $x \geq a > 0$ ) примитивне функције за функције  $f : x \mapsto \frac{1}{\sqrt{x^2 + a^2}}$  и  $g : x \mapsto \frac{1}{\sqrt{x^2 - a^2}}$

### Својства интеграла

$$\int k \cdot f(x)dx = k \cdot \int f(x)dx \quad (\text{хомогеност})$$

$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx \quad (\text{адитивност})$$

$$\int \sum_{i=1}^n c_i f_i(x)dx = \sum_{i=1}^n c_i \int f_i(x)dx \quad (\text{линеарност})$$

Налажење интеграла – Таблица + својства

1.4. [Зад.14, стр.74]

$$f(x) = \frac{(x-1)(x^2+2)}{5x^2}, I = \int f(x)dx = ?$$

1.5. [Зад.15, стр.74]

$$f(x) = \frac{x^2}{1+x^2}, I = \int f(x)dx = ?$$

1.6. [Зад.16, стр.74]

$$f(x) = \tan^2 x, I = \int f(x)dx = ?$$

1.7. [Зад.18, стр.74]

$$f(x) = \frac{1}{\sin^2 x \cos^2 x}, I = \int f(x)dx = ?$$

1.8. [Није из Збирке]

$$f(x) = \frac{1}{a^2 - x^2}, I = \int f(x)dx = ?$$

1.9. [Није из Збирке]

$$f(x) = \frac{1}{(1-x^2)(1+x^2)}, I = \int f(x)dx = ?$$

Замена променљиве

Ако је

$$\int f(x)dx = F(x) + C,$$

тада је

$$\int f(u(x))du(x) = F(u(x)) + C$$

На пример,

$$\text{из } \int \frac{dx}{x} = \ln|x| + C \text{ имамо } \int \frac{du(x)}{u(x)} = \ln|u(x)| + C$$

$$\text{из } \int \frac{dx}{1+x^2} = \arctan x + C \text{ имамо } \int \frac{du(x)}{1+u^2(x)} = \arctan u(x) + C$$

1.10. [Није из Збирке]

$$f(x) = \frac{1}{a^2 + x^2} \quad (a \neq 0), I = \int f(x)dx = ?$$

1.11. [Није из Збирке]

$$f(x) = \frac{1}{\sqrt{a^2 - x^2}} \quad (a > 0), I = \int f(x)dx = ?$$

1.12. [Зад.29, стр.74]

$$f(x) = \tan x, I = \int f(x)dx = ?$$

1.13. [Зад.35, стр.74 ]

$$f(x) = \frac{\sin 2x}{1 + \sin^2 x}, I = \int f(x)dx = ?$$

1.14. [Није из Збирке]

$$I = \int xe^{-x^2} dx = ?$$

1.15. [Није из Збирке]

$$I = \int \frac{1}{x^2} \sin \frac{1}{x} = ?$$

1.16. [Зад.38, стр.74]

$$f(x) = \sin^3 x \cos x dx, I = \int f(x)dx = ?$$

1.17. [Зад.39, стр.74]

$$I = \int \frac{\arctan^2 x}{1 + x^2} dx = ?$$

1.18. [Зад.43, стр.75]

$$I = \int \frac{x dx}{1 + x^4} = ?$$

1.19. [Зад.46, стр.75]

$$I = \int \frac{x^3 dx}{\sqrt{1 - x^8}} = ?$$

1.20. [Зад.47, стр.75]

$$I = \int \frac{dx}{\sqrt{x(1-x)}} = ?$$

1.21. [Зад.45, стр.75]

$$I = \int \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} = ?$$

Метода смене

$x = \varphi(t)$ ,  $\varphi$  - диференцијабилна

$$\int f(x)dx = \int f(\varphi(t))\varphi'(t)dt = \int g(t)dt = G(t) + C = G(\varphi^{-1}(x)) + C$$

$F(x) = G(\varphi^{-1}(x))$  ако постоји  $\varphi^{-1}$

1.22. [Зад.61, стр.75]

$$I = \int \frac{dx}{\sqrt{x}(1 + \sqrt[3]{x})}$$

1.23. [Није из Збирке]

$$I = \int \frac{dx}{(1+x)\sqrt{x}}$$

1.24. [Зад.64, стр.76]

$$I = \int \frac{dx}{x^2\sqrt{1+x^2}}$$

1.25. [Зад.76, стр.76]

$$I = \int \sqrt{a^2 - x^2} dx, \quad a > 0$$

Смена  $\psi(x) = t$

Ако је  $f(x) = g(\psi(x))$ , тада сменом  $\psi(x) = t$  имамо

$$\int f(x) dx = \int g(\psi(x)) dx = \int g(t) d(\psi^{-1}(t)) = \int h(t) dt = H(t) + C$$

Дакле,

$$\int f(x) dx = H(\psi(x)) + C$$

1.26. [Није из Збирке]

$$I = \int e^{e^{-x}-x} dx$$

1.27. [Зад.71, стр.76]

$$I = \int \frac{\cos(\ln x)}{x} dx$$

1.28. [Није из Збирке]

$$I = \int \frac{dx}{1+e^x}$$

1.29. [Зад.74, стр.76]

$$I = \int \frac{1}{1-x^2} \ln \frac{1+x}{1-x} dx$$

1.30. [Зад.75, стр.76]

$$I = \int \frac{\arctan \sqrt{x}}{\sqrt{x}(1+x)} dx$$

Метода парцијалне интеграција

$$d(uv) = u dv + v du$$

$$\int d(uv) = \int u dv + \int v du$$

$$\int u dv = uv - \int v du$$

Корисити се за  $\int x^n \cdot \{\ln x, e^{ax}, \sin ax, \cos ax, \arcsin ax, \arctan ax\} dx$

$$\int e^{ax} \cdot \{\sin bx, \cos bx\} dx$$

1.31. [Зад.90, стр.77]

$$I = \int \frac{x^2 dx}{(1+x^2)^2} = ?$$

1.32. [Није из Збирке]

$$I_2 = \int \frac{dx}{(x^2+a^2)^2} = ?$$

1.33. [Није из Збирке]

$$I = \int \sqrt{a^2+x^2} dx = ?$$

1.34. [Није из Збирке]

$$I = \int \frac{x}{\sqrt{1-x^2}} \ln \frac{1+x}{1-x} dx = ?$$

1.35. [Зад.113, стр.78]

$$I = \int \arcsin^2 x dx = ?$$

1.36. [зад.110, стр.78]

$$I = \int x^2 \arccos x dx = ?$$

1.37. [Колоквијум, 2002 (Збирка, Пр.9)]

$$I_n = \int \arcsin^n x dx, \text{ Веза } I_n \text{ и } I_{n-2} ?$$

## 2 Интеграција рационалних функција

$$f(x) = \frac{P(x)}{Q(x)}, \quad st(P) < st(Q)$$

Ако је

$$Q(x) = a_n(x - a_1)^{k_1}(x - a_2)^{k_2} \cdots (x - a_{k_l})^{k_l}$$

тада сваком фактору  $(x - a)^k$  одговара збир парцијалних разломака

$$\frac{A_1}{x - a} + \frac{A_2}{(x - a)^2} + \cdots + \frac{A_k}{(x - a)^k}$$

2.1. [Није из Збирке]

$$\int \frac{x dx}{(x + 1)(x + 2)(x + 3)}$$

2.2. [зад.132, стр.79]

$$I = \int f(x) dx, \quad f(x) = \frac{x^4 - 3x^2 - 3x - 2}{x^3 - x^2 - 2x}$$

2.3. [Није из Збирке]

$$I = \int \frac{x^2 + 5x - 2}{(x^2 - 1)(x + 1)} dx = ?$$

2.4. [зад.133, стр.79]

$$I = \int \frac{5x - 8}{x^3 - 4x^2 + 4x} dx = ?$$

2.5. [Није из Збирке]

$$I = \int \frac{dx}{x^3(x + 1)^3} = ?$$

Комплексне нуле

$$Q(x) = Q_1(x)(x^2 + bx + c), \quad b^2 - 4ac < 0$$

где је  $Q_1(x)$  као у претходном случају

Триному  $x^2 + bx + c$  одговара сабирак  $\frac{Ax + B}{x^2 + bx + c}$

$\int \frac{Ax + B}{x^2 + bx + c} dx$  се своди на

$$\int \frac{d(x^2 + bx + c)}{x^2 + bx + c} dx = \ln(x^2 + bx + c) + C$$

и

$$\int \frac{dx}{(x - b)^2 + a^2} = \frac{1}{a} \arctan \frac{x - b}{a} + C$$



2.6. [Није из Збирке]

$$\int \frac{3x - 2}{x^2 + 4x + 13} dx = ?$$

2.7. [Није из Збирке]

$$I = \int \frac{x^5 - 1}{x^3 + x^2 + x} dx = ?$$

2.8. [Зад.141, стр.79]

$$I = \int \frac{x dx}{x^3 - 1} = ?$$

2.9. [Није из Збирке]

$$I = \int \frac{dx}{x(1 + x^2)^2} = ?$$

### 3 Интеграција неких тригонометријских функција

Ако је  $f(x) = R(\sin x, \cos x)$ , где је  $R(u, v)$  рационална функција аргумената  $u$  и  $v$ , тада се сменом  $\tan \frac{x}{2} = t$  интеграција функције  $f$  своди на интеграцију рационалних функција.

$$x = 2 \arctan t, \quad dx = \frac{2dt}{1 + t^2},$$

$$\sin x = \frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2}} = \frac{2t}{1 + t^2},$$

$$\cos x = \frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2}} = \frac{1 - t^2}{1 + t^2}$$

$$\int R(\sin x, \cos x) dx = \int R\left(\frac{2t}{1 + t^2}, \frac{1 - t^2}{1 + t^2}\right) \frac{2dt}{1 + t^2}$$

3.1. [Зад.150, стр.80]

$$I = \int \frac{dx}{1 + \sin x + \cos x}$$

3.2. [Зад.154, стр.80]

$$\int \frac{1 - \sin x + \cos x}{1 + \sin x - \cos x} dx$$

Триг. ф. - Специјални случајеви

$$R(u, -v) = -R(u, v), \quad \sin x = t$$

$$\int \frac{\cos x dx}{\sin^2 x - 3 \sin x + 2} = \int \frac{dt}{t^2 - 3t + 2}$$

$$R(-u, v) = -R(u, v), \quad \cos x = t$$

$$\int \frac{\sin x(1 - \cos x)}{\cos x(1 + \cos^2 x)} dx = \int \frac{t-1}{t(1+t^2)} dt$$

$$R(-u, -v) = R(u, v), \quad \tan x = t$$

$$\int \frac{\sin^3 x}{\cos^5 x} dx = \int t^3 dt = \frac{t^4}{4} + C = \frac{1}{4} \tan^4 x + C$$

3.3. [Није из Збирке]

$$I = \int \frac{dx}{4 - 3 \cos^2 x + 5 \sin^2 x}$$

3.4. [Није из Збирке]

$$I = \int \frac{1 - \tan x}{1 + \tan x} dx$$

3.5. [Зад.161, стр.80]

$$I = \int \frac{dx}{1 + \sin^2 x}$$

## 4 Интеграција ирационалних функција

1. Функције облика  $R(x, \sqrt{ax+b})$ ,  $\sqrt{ax+b} = t$

2. Функције облика  $R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right)$ ,  $\sqrt[n]{\frac{ax+b}{cx+d}} = t$

3. Функције облика  $R(x, \sqrt{a^2-x^2})$ ,  $x = a \sin t$

4. Функције облика  $R(x, \sqrt{x^2-a^2})$ ,  $x = a \cosh t$  или  $x = \frac{a}{\sin t}$

5. Функције облика  $R(x, \sqrt{x^2+a^2})$ ,  $x = a \sinh t$  или  $x = a \cdot \tan t$

6. Функције облика  $R(x, \sqrt{ax^2+bx+c})$  - своди се на 3.,4. или 5., а могу и Ојлерове смене:

$$\sqrt{ax^2+bx+c} = t - x\sqrt{a} \text{ за } a > 0$$

$$\sqrt{ax^2+bx+c} = xt - \sqrt{c} \text{ за } c > 0$$

$$\sqrt{ax^2+bx+c} = (x-\alpha)t \text{ за } ax^2+bx+c = a(x-\alpha)(x-\beta)$$

Користе се таблични интегрални:

$$1. \int \frac{dx}{\sqrt{x^2+a^2}} = \ln(x + \sqrt{x^2+a^2}) + C$$

$$2. \int \frac{dx}{\sqrt{x^2-a^2}} = \ln(x + \sqrt{x^2-a^2}) + C$$

$$3. \int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{a} + C$$

а такође и интеграли:

$$4. \int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \ln(x + \sqrt{x^2 + a^2}) + C$$

$$5. \int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln |x + \sqrt{x^2 - a^2}| + C$$

$$6. \int \sqrt{a^2 - x^2} dx = \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2} + C$$

4.1. [Није из Збирке]

$$I = \int \frac{dx}{(x+1)\sqrt{1-x}}$$

4.2. [Није из Збирке]

$$I = \int \frac{1}{(1+x)^2} \sqrt[3]{\frac{1-x}{1+x}} dx$$

4.3. [Зад.167, стр.80]

$$I = \int \frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}} dx$$

4.4. [Зад.169, стр.80]

$$\int \frac{dx}{(1-x)\sqrt{1-x^2}}$$

4.5. [Није из Збирке]

$$I = \int \frac{dx}{\sqrt{1-x^2} - 1}$$

4.6. [Није из Збирке]

$$I = \int \sqrt{(x^2 - 1)^3} dx$$

4.7. [Није из Збирке]

$$I = \int \frac{dx}{x^2 \sqrt{x^2 + 1}}$$

4.8. [Није из Збирке]

$$I = \int \frac{\sqrt{x^2 + 1}}{x^4} dx$$

4.9. [Није из Збирке]

$$I = \int \sqrt{x^2 + 6x} dx$$

4.10. [Зад.172, стр.80]

$$I = \int \frac{5x + 4}{\sqrt{x^2 + 2x + 5}} dx$$

4.11. [Није из Збирке]

$$I = \int \frac{xdx}{\sqrt{x^2 + x + 1}}$$

4.12. [Зад.171, стр.80]

$$I = \int \frac{x^2 dx}{\sqrt{x^2 + x + 1}}$$

4.13. [Зад.174, стр.81]

$$\int \sqrt{4x^2 - 4x + 3} dx$$

4.14. [Није из Збирке]

$$I = \int x^2 \sqrt{4 - x^2} dx$$

4.15. [Зад.170, стр.80]

$$I = \int \frac{dx}{(1+x)\sqrt{1+x-x^2}}$$

## 5 Одређени интеграл

### Услови интеграбилности

Неопходан услов интеграбилности

$$f \in \mathcal{R}[a, b] \Rightarrow f \text{ ограничена на } [a, b]$$

Довољни услови интеграбилности

1.  $f \in C[a, b] \Rightarrow f \in \mathcal{R}[a, b]$

2.  $f$  ограничена и има коначан број прекида на  $[a, b] \Rightarrow f \in \mathcal{R}[a, b]$

На пример, за  $f(x) = \frac{1}{x}$  и  $g(x) = \frac{x}{\sin x}$  имамо

$$f \in \mathcal{R}[1, 2], \quad f \notin \mathcal{R}[-1, 1], \quad g \in \mathcal{R}[-\pi/2, \pi/2], \quad g \notin [\pi/2, 3\pi/2]$$

### Њутн Лајбницова формула

Ако је  $f \in C[a, b]$  и ако је  $F$  примитивна за  $f$  на  $[a, b]$ , тада је

$$\int_a^b f(x) dx = F(b) - F(a) = F(x) \Big|_a^b$$

На пример, (Збирка - зад.32, стр.84 и зад.40, стр.85)

$$1. \int_e^{e^2} \frac{dx}{x \ln x} = \int_e^{e^2} \frac{d(\ln x)}{\ln x} = \ln(\ln x) \Big|_e^{e^2} = \ln(\ln e^2) - \ln(\ln e) = \ln 2$$

$$2. \int_1^e \frac{dx}{x(1 + \ln^2 x)} = \int_1^e \frac{d(\ln x)}{1 + \ln^2 x} = \arctan(\ln x) \Big|_1^e \\ = \arctan(\ln e) - \arctan(\ln 1) = \frac{\pi}{4}$$

Ако је  $F$  примитивна за  $f$  на  $(a, b)$ , тада је

$$\int_a^b f(x)dx = F(b_-) - F(a_+) = F(x) \Big|_{a_+}^{b_-}$$

5.1. [Зад.41, стр.85]

$$I = \int_0^1 \sqrt{4 - x^2} dx$$

5.2. [Зад.45, стр.85]

$$I = \int_0^{\pi/4} \frac{dx}{1 + 2 \sin^2 x}$$

5.3. [Није из Збирке]

$$I = \int_{1/e}^e |\ln x| dx$$

5.4. [Није из Збирке]

$$I = \int_0^\pi \frac{dx}{3 + 2 \cos x}$$

#### Метода смене

$$f \in C[a, b], \quad \varphi \in C^{(1)}[\alpha, \beta], \quad \varphi(\alpha) = a, \quad \varphi(\beta) = b$$

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt$$

• Ако је  $f$  непарна, тада је  $\int_{-a}^a f(x)dx = 0$

На пример,  $\int_{-1/2}^{1/2} \cos x \ln \frac{1+x}{1-x} dx = 0$

• Ако је  $f$  парна, тада је  $\int_{-a}^a f(x)dx = 2 \int_0^a f(x)dx$

5.5. [Није из Збирке]

$$I = \int_0^a x^2 \sqrt{a^2 - x^2} dx, a > 0$$

5.6. [Није из Збирке]

$$I = \int_1^{\sqrt{3}} \frac{x^3 + 1}{x^2 \sqrt{4 - x^2}} dx$$

5.7. [Није из Збирке]

$$I = \int_0^{\sqrt{2}} x^3 \sqrt{16 - x^8} dx$$

5.8. [Није из Збирке]

$$I = \int_4^5 x \sqrt{x^2 - 4x} dx$$

5.9. [Није из Збирке]

$$I = \int_0^{\pi/2} \frac{dx}{1 + \sin x + \cos x}$$

5.10. [Колоквијум, 2002.]

$$\int_0^{\pi/2} \frac{\sin^3 x}{\sin^3 x + \cos^3 x} dx$$

5.11. [Није из Збирке]

$$I = \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$$

5.12. [Није из Збирке]

$$I = \int_{\pi/4}^{\pi/3} \frac{dx}{\sin x \cos^3 x}$$

Метода парцијалне интеграције

$$u, v \in C^{(1)}[a, b] \Rightarrow \int_a^b u dv = uv \Big|_a^b - \int_a^b v du$$

5.13. [Није из Збирке]

$$I = \int_0^{1/2} x \ln \frac{1+x}{1-x} dx$$

5.14. [Зад.74, стр.86]

$$I = \int_0^e x^2 \ln^2 x dx$$

5.15. [Зад.80, стр.87]

$$I = \int_0^3 \arcsin \sqrt{\frac{x}{x+1}} dx$$

5.16. [Није из Збирке]

$$\int_0^1 \frac{x \ln(x + \sqrt{1+x^2})}{(1+x^2)^2} dx$$

5.17. [Зад.83, стр.87]

$$I_n = \int_0^a (a^2 - x^2)^n dx, \quad a > 0$$

## 6 Примена одређеног интеграла

Површина равне фигуре

За фигуру ограничену са  $y = f(x)$ ,  $y = 0$ ,  $x = a$ ,  $x = b$  је

$$P = \int_a^b |f(x)| dx$$

На пример, (зад.85, стр.87)

$$y = \sin x, \quad 0 \leq x \leq \pi \quad P = \int_0^\pi \sin x dx = -\cos x \Big|_0^\pi = -(-1 - 1) = 2$$

За фигуру ограничену са  $y = f(x)$ ,  $y = g(x)$ ,  $x = a$ ,  $x = b$  је

$$P = \int_a^b |f(x) - g(x)| dx$$

На пример, (зад.91, стр.87)

$y = \sin x$ ,  $y = \cos x$ ,  $0 \leq x \leq \pi/4$

$$P = \int_0^{\pi/4} (\cos x - \sin x) dx = \sin x + \cos x \Big|_0^{\pi/4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 0 - 1 = \sqrt{2} - 1$$

6.1. [Зад.99, стр.88]

$$y = \ln(1+x), \quad y = -xe^{-x}, \quad x = 1$$

6.2. [Колоквијум, 2008]

$$y = \frac{1}{\sqrt{x^2 + 2x + 2}}, \quad y = 1, \quad x = 0, \quad P = ?$$

6.3. [Није из Збирке]

$$\sqrt{y} + \sqrt{x} = a \quad (a > 0), \quad x = 0, \quad y = 0$$

6.4. [Није из Збирке]

$$y = x - 1, \quad y^2 = x + 1$$

Дужина лука криве линије

$$y = f(x), \quad x \in [a, b] \quad l = \int_a^b \sqrt{1 + f'^2(x)} dx$$

$$x = g(y), y \in [c, d] \quad l = \int_c^d \sqrt{1 + g'^2(y)} dy$$

$$x = \varphi(t), y = \psi(t), t \in [t_1, t_2] \quad l = \int_{t_1}^{t_2} \sqrt{\varphi'^2(t) + \psi'^2(t)} dt$$

$$\rho = \rho(\varphi), \varphi \in [\alpha, \beta] \quad l = \int_\alpha^\beta \sqrt{\rho^2 + \rho'^2} d\varphi$$

6.5. [Зад.112, стр.89]

$$f(x) = \frac{x^2}{2} - \frac{\ln x}{4}, x \in [1, 3]$$

6.6. [Зад.115, стр.89]

$$y = \ln(x^2 - 1), \quad 2 \leq x \leq 5$$

6.7. [Зад.117, стр.89]

$$y = \ln(\sin x), \quad \frac{\pi}{3} \leq x \leq \frac{2\pi}{3}$$

6.8. [Није из Збирке]

$$y = \ln \frac{e^x - 1}{e^x + 1}, \quad x \in [1, 2]$$

6.9. [Није из Збирке]

$$y = \frac{x^2}{2} - 1 \text{ испод } x\text{-осе}$$

6.10. [Колоквијум, 2003, а и 2006]

$$y = \sqrt{x^2 - 48} + 4\sqrt{6} \ln(x + \sqrt{x^2 - 48}), \quad 7 \leq x \leq 8$$

6.11. [Зад.123, стр.89]

$$x(t) = a \cos^3 t, y(t) = a \sin^3 t \text{ (Астроида)}$$

6.12. [Зад.124, стр.89]

$$x = a(t - \sin t), y = a(1 - \cos t), t \in [0, 2\pi] \text{ (Циклоида)}$$

6.13. [Колоквијум, 2002]

$$x(t) = 2\sqrt{2}\sqrt{1 - t^2}, y(t) = t\sqrt{1 - t^2}, \quad t \in [0, 1]$$

Запремина ротационог тела

$$y = f(x), x \in [a, b] \quad V_x = \pi \int_a^b f^2(x) dx, \quad V_y = 2\pi \int_a^b |xf(x)| dx$$



$$x = g(y), y \in [c, d] \quad V_y = \pi \int_c^d g^2(y) dy, \quad V_x = 2\pi \int_c^d |yg(y)| dy$$

$$x = \varphi(t), y = \psi(t), t \in [\alpha, \beta] \quad V_x = \pi \int_\alpha^\beta \psi^2(t) \varphi'(t) dt$$

$$\rho = \rho(\theta), \theta \in [\alpha, \beta] \quad V = \frac{2\pi}{3} \int_\alpha^\beta \rho^3(\theta) \sin \theta d\theta$$

6.14. [Није из Збирке]

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad V_x = ?, \quad V_y = ?$$

6.15. [Није из Збирке]

$$y = \sin x, x = 0, y = 1 \quad V_y = ?$$

6.16. [Није из Збирке]

$$y = e^x, x = 0, x = 2, y = 0, \quad V_y = ?$$

6.17. [Зад.146, стр.90]

$$y^2 = 2x, y = 2, x = 0, \quad V_x = ?$$

6.18. [Зад.147, стр.90]

$$y = x, y = \frac{1}{x}, x = 3, \quad V_x = ?$$

6.19. [Зад.155, стр.91]

$$y = 2x - x^2, y = 0, \quad V_y = ?$$

6.20. [Није из Збирке]

$$y = 2x - x^2, y = 0, \quad V_x = ?$$

6.21. [Није из Збирке]

$$x^2 + y^2 = r^2, (x - r)^2 + y^2 = r^2, \quad V = ? \text{ (заједнички део)}$$

6.22. [Није из Збирке]

$$y = x^2, 8x = y^2, \quad V_y = ?$$

Површина омотача рот. тела

$$y = f(x), x \in [a, b] \quad P_x = 2\pi \int_a^b y \sqrt{1 + y'^2} dx$$

$$x = g(y), y \in [c, d] \quad P_y = 2\pi \int_c^d x \sqrt{1 + x'^2} dy$$

$$x = \varphi(t), y = \psi(t), t \in [\alpha, \beta] \quad P_x = 2\pi \int_{\alpha}^{\beta} \psi(t) \sqrt{\varphi'^2(t) + \psi'^2(t)} dt$$

$$\rho = \rho(\theta), \theta \in [\alpha, \beta] \quad P = 2\pi \int_{\alpha}^{\beta} \rho(\theta) \sqrt{\rho^2(\theta) + \rho'^2(\theta)} \sin \theta d\theta$$

6.23. [Збирка, зад.160, стр.91]

$$y = x^3, 0 \leq x \leq 1, \text{ око } x\text{-осе}$$

6.24. [Колоквијум, 2009]

$$y = x^2 - \frac{\ln x}{8}, \sqrt{e} \leq x \leq e, \text{ око } x\text{-осе}$$

6.25. [Није из Збирке]

$$y = x^2, x = y^2, V_x = ?, P_x = ?$$

6.26. [Није из Збирке]

$$y = \sin x, x \in [0, \pi]$$

## 7 Несвојствени интеграли

???????????

## 8 Двојни интеграли

Декартове координате

$$\mathcal{D} = [a, b] \times [c, d]$$

$$\iint_{\mathcal{D}} f(x, y) dx dy = \int_a^b dx \int_c^d f(x, y) dy = \int_c^d dy \int_a^b f(x, y) dx$$

$$\mathcal{D} = \{(x, y) \mid x \in [a, b], y_1(x) \leq y \leq y_2(x)\}$$

$$\iint_{\mathcal{D}} f(x, y) dx dy = \int_a^b dx \int_{y_1(x)}^{y_2(x)} f(x, y) dy$$

$$\mathcal{D} = \{(x, y) \mid y \in [c, d], x_1(y) \leq x \leq x_2(y)\}$$

$$\iint_{\mathcal{D}} f(x, y) dx dy = \int_c^d dy \int_{x_1(y)}^{x_2(y)} f(x, y) dx$$

8.1. [Није из Збирке]

$$\iint_{\mathcal{D}} \frac{x^2}{1+y^2} dx dy, \quad \mathcal{D} = [0, 1] \times [0, 1]$$

8.2. [Зад.18, стр.99]

$$\iint_{\mathcal{D}} (x^3 + y^3) dx dy, \quad \mathcal{D} \text{ је област ограничена правама } y = x/2, y = x \text{ и } x = 4$$

8.3. [Није из Збирке]

$$\iint_{\mathcal{D}} \frac{dx dy}{(x + y + 1)^2}, \quad \mathcal{D} = [0, 1] \times [0, 1]$$

8.4. [Није из Збирке]

$$\iint_{\mathcal{D}} (x + y) dx dy, \quad \mathcal{D} : y^2 = 2x, x^2 = 2y$$

8.5. [Није из Збирке]

$$\iint_{\mathcal{D}} (x^2 + y^2) dx dy, \quad \mathcal{D} : y = x, y = x + 1, y = 1, y = 3$$

8.6. [Није из Збирке]

$$\iint_{\mathcal{D}} xy dx dy, \quad \mathcal{D} : x^2 + y^2 = 1, x^2 + y^2 = 2x, y = 0$$

8.7. [Зад.25, стр.99]

$$\iint_{\mathcal{D}} x^2 y^2 \sqrt{1 - x^3 - y^3} dx dy, \quad \mathcal{D} = \{(x, y) : x \geq 0, y \geq 0, x^3 + y^3 \leq 1\}$$

8.8. [Колоквијум, 2002]

$$I = \iint_{\mathcal{D}} x \cdot \sin |y - x^2| dx dy, \quad \mathcal{D} = \{(x, y) : 0 \leq x \leq \sqrt{\pi/2}, 0 \leq y \leq \pi\}$$

Смене променљивих

$$F : \mathcal{G} \rightarrow \mathcal{D}, \quad F : (u, v) \mapsto (x, y)$$

$$x = x(u, v), \quad y = y(u, v), \quad dx dy = |J| du dv, \quad J = \frac{D(x, y)}{D(u, v)} = \begin{vmatrix} x'_u & x'_v \\ y'_u & y'_v \end{vmatrix}$$

$$\iint_{\mathcal{D}} f(x, y) dx dy = \iint_{\mathcal{G}} f(x(u, v), y(u, v)) |J| du dv$$

Важи

$$J = \frac{D(x, y)}{D(u, v)} = \frac{1}{\frac{D(u, v)}{D(x, y)}}$$

8.9. [Није из Збирке]

$$\iint_{\mathcal{D}} (x^2 - y^2) dx dy, \quad \mathcal{D} : x + y = 1, x + y = 3, 2x - y = 2, 2x - y = -1$$

8.10. [Није из Збирке]

$$\iint_{\mathcal{D}} \cos \frac{x - y}{x + y} dx dy, \quad \mathcal{D} : x + y = 1, x = 0, y = 0$$

8.11. [Није из Збирке]

$$\iint_{\mathcal{D}} \sin \frac{\pi xy}{a^2} dx dy, \quad \mathcal{D} : y = \alpha x, y = \beta x, xy = a, x > 0, a > 0, \beta > \alpha > 0$$

Поларне координате

$$x = \rho \cos \varphi, y = \rho \sin \varphi, \quad J = \rho$$

$$\text{За } x^2 + y^2 \leq R^2 \text{ је } (\rho, \varphi) \in [0, R] \times [0, 2\pi]$$

Елиптичке координате

$$x = a\rho \cos \theta, y = b\rho \sin \theta, \quad dx dy = ab\rho d\rho d\theta, \quad J = ab\rho$$

8.12. [Зад.35, стр.100]

$$\iint_{\mathcal{D}} \sqrt{a^2 - x^2 - y^2} dx dy, \quad \mathcal{D} = \{(x, y) \mid x^2 + y^2 \leq a^2, x \geq 0, y \geq 0\}$$

8.13. [Зад.36, стр.100]

$$\iint_{\mathcal{D}} \frac{\sin \sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}} dx dy, \quad \mathcal{D} = \{(x, y) \mid \pi^2/9 \leq x^2 + y^2 \leq \pi^2\}$$

8.14. [Није из Збирке]

$$I = \iint_{\mathcal{D}} (1 - 2x - 3y) dx dy, \quad \mathcal{D} : x^2 + y^2 \leq R$$

8.15. [Колоквијум, 2004]

$$\iint_{\mathcal{D}} \frac{dx dy}{(x^2 + y^2)^2}, \quad \mathcal{D} : x^2 + y^2 = 4x, x^2 + y^2 = 8x, y = 0, y = x$$

8.16. [Зад.37, стр.100]

$$I = \iint_{\mathcal{D}} \sqrt{R^2 - x^2 - y^2} dx dy, \quad \mathcal{D} : x^2 + y^2 \leq Rx, y \geq 0$$

8.17. [Зад.38, стр.100]

$$I = \iint_{\mathcal{D}} \arctan \frac{y}{x} dx dy, \quad \mathcal{D} : x^2 + y^2 = 1, x^2 + y^2 = 9, y = \frac{\sqrt{3}}{3}x, y = \sqrt{3}x$$

8.18. [Није из Збирке]

$$\iint_{\mathcal{D}} \sqrt{4 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} dx dy, \quad \mathcal{D} : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \frac{x^2}{4a^2} + \frac{y^2}{4b^2} = 1, x \geq 0, y \geq 0$$

## 9 Примена двојног интеграла

### Запремине тела

Запремина цилиндричног тела чија је једна основа фигура  $\mathcal{D}$  у равни  $Oxy$ , а друга основа површ  $z = f(x, y)$

$$V = \iint_{\mathcal{D}} f(x, y) dx dy$$

9.1. [Зад.48, стр.101]

$$z = 4x^2 + 2y^2 + 1, \quad x + y - 3 = 0, \quad x = 0, \quad y = 0, \quad z = 0$$

9.2. [Зад.53, стр.101]

$$z = xy, \quad z = 0, \quad (x - 1)^2 + (y - 1)^2 = 1$$

9.3. [Зад.57, стр.101]

$$z = a^2 - x^2, \quad x^2 + y^2 = a^2, \quad x = 0, \quad y = 0, \quad z = 0$$

9.4. [Зад.62, стр.101]

$$x^2 + y^2 = 2x, \quad x^2 + y^2 = 2y, \quad z = x + 2y, \quad z = 0$$

9.5. [Није из Збирке]

$$T : z = x^2 + y^2, \quad z = 2(x^2 + y^2), \quad y = x, \quad y^2 = x, \quad V = ?$$

9.6. [Колоквијум, 2003]

$$T : x^2 + y^2 = 2x, \quad z = 2x + y, \quad z = 0, \quad y \geq 0, \quad V = ?$$

9.7. [Колоквијум, ??]

$$T : z = 0, \quad z = \frac{x^2 + y^2}{2}, \quad (x - 1)^2 + y^2 = 1, \quad V = ?$$

### Површине равних фигура

Површину  $P(\mathcal{D})$  фигуре  $\mathcal{D}$  можемо добити ако у двојном интегралу на области  $\mathcal{D}$  узмемо функцију  $f(x, y) = 1$

$$P(\mathcal{D}) = \iint_{\mathcal{D}} dx dy$$

9.8. [Зад.73, стр.102]

$$y = \frac{1}{a}(x - a)^2, \quad x^2 + y^2 = a^2, \quad a > 0$$

9.9. [Зад.74, стр.102]

$$x^2 + y^2 = a^2, \quad x + y = a, \quad y = \frac{a}{2}, \quad x \geq 0, \quad y \geq \frac{a}{2}, \quad a > 0$$

9.10. [Зад.75, стр.102]

$$xy = a^2, \quad x^2 = ay, \quad y = 2a, \quad x = 0, \quad a > 0$$

9.11. [Зад.79, стр.102]

$$x = \frac{y^2 + b^2}{2b}, \quad x = \frac{y^2 + a^2}{2a}, \quad a > b > 0$$

9.12. [Зад.77, стр.102]

$$(x - a)^2 + y^2 = a^2, \quad x^2 + (y - a)^2 = a^2$$

9.13. [Зад.78, стр.102]

$$x^2 + y^2 - 2ax = 0, \quad x^2 + y^2 - ax = 0$$

9.14. [Није из Збирке]

$$\mathcal{D} : x^2 + y^2 \leq 2x, \quad |y| \leq x, \quad P(\mathcal{D}) = ?$$

9.15. [Није из Збирке]

$$\mathcal{D} : \left(\frac{x}{a} + \frac{y}{b}\right)^2 = \frac{x}{a} - \frac{y}{b}, \quad y = 0, \quad P(\mathcal{D}) = ?$$

Површина дела површи

Површина дела површи  $z(x, y)$  над  $\mathcal{D}$  је дата са

$$P = \iint_D \sqrt{1 + z_x'^2 + z_y'^2} dx dy$$

9.16. [Није из Збирке]

$$z = x^2 + y^2, \quad \mathcal{D} : x^2 + y^2 \leq 1, \quad P = ?$$

9.17. [Није из Збирке]

$$x^2 + y^2 + x^2 = a^2, \quad \mathcal{D} : x^2 + y^2 = ax$$

9.18. [Зад.88, стр.103]

Израчунати површину дела конуса  $x^2 + y^2 = z^2$  који исеца цилиндар  $x^2 + y^2 = 2ax$ .

## 10 Тројни интеграли

## РЕШЕЊА

1.1. Једна примитивна је  $\arctan x$

Скуп свих примитивних је  $\{\arctan x + C, C \in \mathbb{R}\}$

Из услова  $\arctan 1 + C = \pi$  следи  $C = \frac{3\pi}{4}$

$$F(x) = \arctan x + \frac{3\pi}{4}$$

1.2.

$$e^{|x|} = \begin{cases} e^x, & x \geq 0 \\ e^{-x}, & x < 0 \end{cases}$$

За  $x > 0$  примитивне су  $e^x + A$ ,  $A \in \mathbb{R}$

За  $x < 0$  примитивне су  $-e^{-x} + B$ ,  $B \in \mathbb{R}$

За  $x \in \mathbb{R}$  примитивна функција  $F$  мора бити диференцијабилна (дакле, и непрекидна) у тачки  $x = 0$

На пример,

$$F(x) = \begin{cases} e^x - 1, & x \geq 0 \\ -e^{-x} + 1, & x < 0 \end{cases}$$

1.3.

$$F'(x) = \frac{1 + \frac{2x}{2\sqrt{x^2+a^2}}}{x + \sqrt{x^2+a^2}} = \frac{\sqrt{x^2+a^2} + x}{\sqrt{x^2+a^2}} \cdot \frac{1}{x + \sqrt{x^2+a^2}} = \frac{1}{\sqrt{x^2+a^2}} = f(x)$$

Слично је и  $G'(x) = g(x)$

Уврстити у табличне  $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln(x + \sqrt{x^2 \pm a^2})$

1.4.

$$f(x) = \frac{1}{5x^2}(x^3 - x^2 + 2x - 2) = \frac{1}{5}x - \frac{1}{5} + \frac{2}{5} \cdot \frac{1}{x} - \frac{2}{5} \cdot \frac{1}{x^2}$$

За  $\alpha \neq -1$  је  $\int x^\alpha dx = \frac{1}{\alpha+1}x^{\alpha+1} + C$ ,  $\int \frac{dx}{x} = \ln|x| + C$

$$\begin{aligned} I &= \frac{1}{5} \cdot \frac{1}{2}x^2 - \frac{1}{5}x + \frac{2}{5} \ln|x| - \frac{2}{5} \cdot \frac{-1}{x} + C \\ &= \frac{1}{10}x^2 - \frac{1}{5}x + \frac{2}{5} \ln|x| + \frac{2}{5} \cdot \frac{1}{x} + C \end{aligned}$$

1.5.

$$f(x) = \frac{x^2 + 1 - 1}{1 + x^2} = \frac{x^2 + 1}{1 + x^2} - \frac{1}{1 + x^2} = 1 - \frac{1}{1 + x^2}$$

$$\int \frac{dx}{1 + x^2} = \arctan x + C$$

$$I = x - \arctan x + C$$

---

1.6.

$$f(x) = \frac{\sin^2 x}{\cos^2 x} = \frac{1 - \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} - 1$$

$$\int \frac{dx}{\cos^2 x} = \tan x + C$$

$$I = \tan x - x + C$$

---

1.7.

$$f(x) = \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} = \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}$$

$$\int \frac{dx}{\sin^2 x} = -\cot x + C$$

$$I = \tan x - \cot x + C$$

---

1.8.

$$\begin{aligned} f(x) &= \frac{1}{(a-x)(a+x)} = \frac{1}{2a} \cdot \frac{a+x+a-x}{(a-x)(a+x)} \\ &= \frac{1}{2a} \cdot \frac{1}{a-x} + \frac{1}{2a} \cdot \frac{dx}{a+x} \\ &= \frac{1}{2a} \cdot \frac{1}{a+x} - \frac{1}{2a} \cdot \frac{dx}{x-a} \end{aligned}$$

$$I = \frac{1}{2a} \int \frac{dx}{a+x} - \frac{1}{2a} \int \frac{dx}{x-a} = \frac{1}{2a} \ln \left| \frac{x+a}{x-a} \right| + C$$

Уврстити у табличне  $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{x+a}{x-a} \right| + C$

---

1.9.

$$f(x) = \frac{1}{2} \cdot \frac{1+x^2+1-x^2}{(1-x^2)(1+x^2)} = \frac{1}{2} \cdot \frac{1}{1-x^2} + \frac{1}{2} \cdot \frac{1}{1+x^2}$$

$$I = \frac{1}{2} \int \frac{dx}{1-x^2} + \frac{1}{2} \int \frac{dx}{1+x^2} = \frac{1}{2} \ln \left| \frac{x+1}{x-1} \right| + \frac{1}{2} \arctan x + C$$

---

1.10.

$$I = \frac{1}{a} \int \frac{d(x/a)}{1+(x/a)^2} = \frac{1}{a} \int \frac{du}{1+u^2} = \frac{1}{a} \arctan u + C$$

Дакле,  $I = \frac{1}{a} \arctan \frac{x}{a} + C$

Уврстити у табличне  $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a} + C$

---



1.11.

$$I = \frac{1}{a} \int \frac{d(x)}{\sqrt{1 - (x/a)^2}} = \int \frac{d(x/a)}{\sqrt{1 - (x/a)^2}} = \int \frac{du}{\sqrt{1 - u^2}} = \arcsin u + C$$

Дакле,  $I = \arcsin \frac{x}{a} + C$

Уврстити у табличне  $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C$

---

1.12.

$$I = \int \frac{\sin x}{\cos x dx} = - \int \frac{d(\cos x)}{\cos x} = - \int \frac{du(x)}{u(x)}, \quad u(x) = \cos x$$

$$I = - \int \frac{du}{u} = - \ln |u| + C = - \ln |\cos x| + C$$

---

1.13.

$$\int f(x) dx = \int \frac{2 \sin x \cos x}{1 + \sin^2 x} dx = \int \frac{d(1 + \sin^2 x)}{1 + \sin^2 x} = \int \frac{du}{u}$$

$$I = \int \frac{du}{u} = \ln |u| + C = \ln(1 + \sin^2 x) + C$$

---

1.14.

$$I = -\frac{1}{2} \int e^{-x^2} d(-x^2) = -\frac{1}{2} \int e^u du = -\frac{1}{2} e^u + C = -\frac{1}{2} e^{-x^2} + C$$

---

1.15.

$$I = - \int \sin \frac{1}{x} d\left(\frac{1}{x}\right) = - \int \sin u du = \cos u + C = \cos \frac{1}{x} + C$$

---

1.16.  $\int f(x) dx = \int \sin^3 x d(\sin x) = \int u^3 du$

$$I = \int u^3 du = \frac{1}{4} u^4 + C = \frac{\sin^4 x}{4} + C$$

---

1.17.  $I = \int \arctan^2 x d(\arctan x) = \int u^2 du = \frac{1}{3} u^3 + C = \frac{1}{3} \arctan^3 x + C$

---

1.18.  $I = \frac{1}{2} \int \frac{d(x^2)}{1 + (x^2)^2} = \frac{1}{2} \int \frac{du}{1 + u^2} = \frac{1}{2} \arctan u + C = \frac{1}{2} \arctan x^2 + C$

---

1.19.  $I = \frac{1}{4} \int \frac{d(x^4)}{\sqrt{1 - (x^4)^2}} = \frac{1}{4} \int \frac{du}{\sqrt{1 - u^2}} = \frac{1}{4} \arcsin u + C = \frac{1}{4} \arcsin x^4 + C$

---

1.20.  $I = \int \frac{dx}{\sqrt{x}\sqrt{1-x}} = 2 \int \frac{d(\sqrt{x})}{\sqrt{1-(\sqrt{x})^2}} = 2 \int \frac{du}{\sqrt{1-u^2}}$

$$I = 2 \arcsin u + C = 2 \arcsin \sqrt{x} + C$$

---

$$1.21. I = \int \frac{dx}{\cos^2 x (a^2 + b^2 \tan^2 x)} = \frac{1}{a^2} \int \frac{d(\tan x)}{1 + \left(\frac{b}{a} \tan x\right)^2} = \frac{1}{a^2} \cdot \frac{a}{b} \int \frac{d\left(\frac{b}{a} \tan x\right)}{1 + \left(\frac{b}{a} \tan x\right)^2}$$

$$I = \frac{1}{ab} \int \frac{du}{1 + u^2} = \frac{1}{ab} \arctan u + C = \frac{1}{ab} \arctan \left( \frac{b \tan x}{a} \right) + C$$


---

$$1.22. x = t^6 = \varphi(t), \sqrt{x} = t^3, \sqrt[3]{x} = t^2, dx = 6t^5 dt, \varphi^{-1} : x \mapsto \sqrt[6]{x}$$

$$I = \int \frac{6t^5 dt}{t^3(1+t^2)} = 6 \int \frac{t^2 dt}{1+t^2} = 6 \int \frac{1+t^2}{1+t^2} dt - 6 \int \frac{dt}{1+t^2}$$

$$I = 6t - 6 \arctan t + C = 6(\sqrt[6]{x} - \arctan \sqrt[6]{x}) + C$$


---

$$1.23. x = t^2, dx = 2t dt, t = \sqrt{x}$$

$$I = \int \frac{2t dt}{(1+t^2)t} = 2 \int \frac{dt}{1+t^2} = 2 \arctan t + A = 2 \arctan \sqrt{x} + A = F(x) + A$$

$$x = 1/t^2, dx = -2dt/t^3, t = 1/\sqrt{x} \text{ (друга смена)}$$

$$I = -2 \int \frac{dt}{1+t^2} = -2 \arctan t + B = -2 \arctan \frac{1}{\sqrt{x}} + B = G(x) + B$$

$$F(x) = G(x) + C, \text{ односно } 2 \arctan \sqrt{x} = -2 \arctan \frac{1}{\sqrt{x}} + C$$

$$\text{За } x = 1 : 2 \cdot \frac{\pi}{4} = -2 \cdot \frac{\pi}{4} + C, C = \pi \text{ Дакле, важи једнакост}$$

$$\arctan \sqrt{x} + \arctan \frac{1}{\sqrt{x}} = \frac{\pi}{2}$$


---

$$1.24. x = 1/t, dx = -dt/t^2, 1 + x^2 = 1 + 1/t^2 = \frac{t^2 + 1}{t^2}$$

$$I = - \int \frac{dt}{t^2} \cdot \frac{t^2 \cdot t}{\sqrt{1+t^2}} = - \int \frac{t dt}{\sqrt{1+t^2}}$$

$$I = - \frac{1}{2} \int \frac{d(1+t^2)}{\sqrt{1+t^2}} = -\sqrt{1+t^2} + C = -\sqrt{1+1/x^2} + C$$


---

$$1.25. x = a \sin t = \varphi(t), \varphi - \text{ на } (-\pi/2, \pi/2) \text{ има инверзну (монотона је)}$$

$$\varphi^{-1} : x \mapsto t = \arcsin \frac{x}{a}$$

$$I = \int \sqrt{a^2 - a^2 \sin^2 t} \cdot a \cos t dt = a^2 \int \cos^2 t dt = \frac{a^2}{2} \int (1 + \cos 2t) dt$$

$$I = \frac{a^2}{2} \left( t + \frac{1}{2} \sin 2t \right) + C = \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2} + C$$


---

$$1.26. \psi(x) = e^{-x} = t, d(e^{-x}) = dt, -e^{-x} dx = dt$$

$$I = \int e^{e^{-x}} \cdot \underbrace{e^{-x} dx}_{-dt} = - \int e^t dt = -e^t + C$$

$$I = -e^{e^{-x}} + C$$


---

$$1.27. \psi(x) = \ln x = t, dx/x = dt$$

$$I = \int \cos(\ln x) \frac{dx}{x} = \int \cos t dt = \sin t + C = \sin(\ln x) + C$$

---

1.28.  $\psi(x) = 1 + e^x = t$ ,  $e^x = t - 1$ ,  $x = \ln(t - 1)$ ,  $dx = \frac{dt}{t - 1}$

$$I = \int \frac{dt}{t(t-1)} = \int \frac{t - (t-1)}{t(t-1)} dt = \int \frac{dt}{t-1} - \int \frac{dt}{t} = \ln|t-1| - \ln|t| + C$$

$$I = \ln \frac{e^x}{1 + e^x} + C$$

---

1.29.  $\ln \frac{1+x}{1-x} = t$ ,  $d\left(\ln \frac{1+x}{1-x}\right) = dt$ ,  $\frac{1-x}{1+x} \cdot \frac{1-x+(1+x)}{(1-x)^2} dx = dt$ ,  $\frac{2dx}{1-x^2} = dt$

$$I = \frac{1}{2} \int t dt = \frac{1}{2} \cdot \frac{t^2}{2} = \frac{1}{4} \ln^2 \frac{1+x}{1-x} + C$$

---

1.30.  $\arctan \sqrt{x} = t$ ,  $d(\arctan \sqrt{x}) = dt$ ,  $\frac{1}{1+x} \cdot \frac{dx}{2\sqrt{x}} = dt$

$$I = \int t \cdot 2dt = t^2 + C = \arctan^2 \sqrt{x} + C$$

---

1.31.  $u = x$  и  $dv = \frac{xdx}{(1+x^2)^2}$

$$du = dx \text{ и } v = -\frac{1}{2} \cdot \frac{1}{1+x^2},$$

$$I = -\frac{x}{2(1+x^2)} + \frac{1}{2} \int \frac{dx}{1+x^2} = -\frac{x}{2(1+x^2)} + \frac{1}{2} \arctan x + C.$$

---

1.32.  $I_1 = \int \frac{dx}{x^2 + a^2}$

Ако за интеграл  $I_1$  применимо парцијалну интеграцију ( $dv = dx$ ), добијамо

$$I_1 = \frac{x}{x^2 + a^2} + 2 \int \frac{x^2 dx}{(x^2 + a^2)^2} = \frac{x}{x^2 + a^2} + 2I_1 - 2a^2 I_2$$

$$I_2 = \frac{1}{2a^2} \cdot \frac{x}{x^2 + a^2} + \frac{1}{2a^2} I_1 = \frac{1}{2a^2} \cdot \frac{x}{x^2 + a^2} + \frac{1}{2a^3} \arctan \frac{x}{a} + C$$

---

1.33.  $u = \sqrt{a^2 + x^2}$ ,  $dv = dx$

$$du = \frac{x}{\sqrt{a^2 + x^2}}, \quad v = x$$

$$\begin{aligned} I &= x\sqrt{a^2 + x^2} - \int \frac{x^2 dx}{\sqrt{a^2 + x^2}} = x\sqrt{a^2 + x^2} - \int \frac{a^2 + x^2 - a^2}{\sqrt{a^2 + x^2}} dx \\ &= x\sqrt{a^2 + x^2} - I + a^2 \ln|x + \sqrt{a^2 + x^2}| + C \end{aligned}$$

$$I = \frac{x}{2} \sqrt{a^2 + x^2} + \frac{a^2}{2} \ln|x + \sqrt{a^2 + x^2}| + C$$

---

1.34.  $u = \ln \frac{1+x}{1-x}, dv = \frac{xdx}{\sqrt{1-x^2}}$

$$du = \frac{1-x}{1+x} \cdot \frac{1-x+1+x}{(1-x)^2} dx, \quad v = -\frac{1}{2} \int \frac{d(1-x^2)}{\sqrt{1-x^2}} = -\sqrt{1-x^2}$$

$$I = -\sqrt{1-x^2} \ln \frac{1+x}{1-x} + 2 \int \frac{dx}{\sqrt{1-x^2}} = -\sqrt{1-x^2} \ln \frac{1+x}{1-x} + 2 \arcsin x + C$$


---

1.35.  $u = \arcsin^2 x, dv = dx, du = 2 \frac{\arcsin x}{\sqrt{1-x^2}} dx, v = x$

$$I = x \cdot \arcsin^2 x - 2J, \quad J = \int \frac{x \arcsin x}{\sqrt{1-x^2}} dx$$

За  $J$ :  $u = \arcsin x$  и  $dv = \frac{xdx}{\sqrt{1-x^2}}$

$$du = \frac{dx}{\sqrt{1-x^2}} \text{ и } v = -\sqrt{1-x^2}$$

$$J = -\sqrt{1-x^2} \arcsin x + x + C$$

$$I = x \cdot \arcsin^2 x + 2\sqrt{1-x^2} \arcsin x - 2x + C$$


---

1.36.  $u = x^2$  и  $dv = \arccos x dx$

$$v = \int \arccos x dx = x \arccos x + \int \frac{xdx}{\sqrt{1-x^2}} = x \arccos x - \sqrt{1-x^2}$$

$$I = x^3 \arccos x - x^2 \sqrt{1-x^2} - 2I + 2 \int x \sqrt{1-x^2} dx.$$

$$3I = x^3 \arccos x - x^2 \sqrt{1-x^2} + 2 \int x \sqrt{1-x^2} dx$$

$$3I = x^3 \arccos x - x^2 \sqrt{1-x^2} - \frac{1}{3} \sqrt{(1-x^2)^3}$$


---

1.37.  $u = \arcsin^n x, dv = dx$

$$I_n = x \cdot \arcsin^n x - n \int \frac{x \cdot \arcsin^{n-1} x}{\sqrt{1-x^2}} dx = x \cdot \arcsin^n x - n \cdot J$$

За  $J$ :  $u = \arcsin^{n-1} x, dv = \frac{x}{\sqrt{1-x^2}}, v = -\sqrt{1-x^2}$

$$J = -\sqrt{1-x^2} \cdot \arcsin^{n-1} x + (n-1)I_{n-2}$$

$$I_n = x \cdot \arcsin^n x + n\sqrt{1-x^2} \cdot \arcsin^{n-1} x - n(n-1)I_{n-2}$$


---

2.1.

$$\frac{x}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$$

$$x = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$$

За  $x = -1$  следи  $A = -1/2$

За  $x = -2$  следи  $B = -2$

За  $x = -3$  следи  $C = -3/2$

$$\begin{aligned}
I &= -\frac{1}{2} \int \frac{dx}{x+1} + 2 \int \frac{dx}{x+2} - \frac{3}{2} \int \frac{dx}{x+3} \\
&= -\frac{1}{2} \ln|x+1| + 2 \ln|x+2| - \frac{3}{2} \ln|x+3| + C \\
&= \ln \frac{(x+2)^2}{|x+1|^{1/2}|x+3|^{3/2}} + C
\end{aligned}$$


---

**2.2.**

$$f(x) = \frac{x^4 - x^3 - 2x^2 + x^3 - x^2 - 2x - x - 2}{x^3 - x^2 - 2x} = x + 1 - \frac{x+2}{x(x^2 - x - 2)}$$

$$\frac{x+2}{x(x-2)(x+1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1}$$

$$x+2 = A(x-2)(x+1) + Bx(x+1) + Cx(x-2)$$

3a  $x = 0$ ,  $2 = A \cdot (-2) \cdot 1$ ,  $A = -1$

3a  $x = -1$ ,  $1 = C \cdot (-1) \cdot (-3)$ ,  $C = 1/3$

3a  $x = 2$ ,  $4 = B \cdot 2 \cdot 3$ ,  $B = 2/3$

$$\begin{aligned}
I &= \frac{x^2}{2} + x + \int \frac{dx}{x} - \frac{2}{3} \int \frac{dx}{x-2} - \frac{1}{3} \int \frac{dx}{x+1} \\
I &= \frac{x^2}{2} + x + \ln|x| - \frac{2}{3} \ln|x-2| - \frac{1}{3} \ln|x+1| + C
\end{aligned}$$


---

**2.3.**

$$\frac{x^2 + 5x - 2}{(x^2 - 1)(x+1)} = \frac{x^2 + 5x - 2}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{(x+1)^2} + \frac{C}{x+1}$$

3a  $x = 1$  je  $1^2 + 5 \cdot 1 - 2 = A \cdot (1+1)^2$ ,  $4A = 4$ ,  $A = 1$

3a  $x = -1$  je  $(-1)^2 + 5 \cdot (-1) - 2 = B \cdot (-1-1)$ ,  $B = 3$

3a  $x = 0$  je  $-2 = A - B - C$ ,  $-2 = 1 - 3 - C$ ,  $C = 0$

$$I = \int \frac{dx}{x-1} + 3 \cdot \int \frac{dx}{(x+1)^2} = \ln|x-1| - \frac{3}{x+1} + C$$


---

**2.4.**  $\frac{5x-8}{x^3-4x^2+4x} = \frac{5x-8}{x(x-2)^2} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$

$$5x-8 = A(x-2)^2 + Bx(x-2) + Cx$$

3a  $x = 0$ ,  $-8 = A \cdot 4$ ,  $A = -2$ ; 3a  $x = 2$ ,  $2 = C \cdot 2$ ,  $C = 1$

$$5x-8 = -2(x-2)^2 + Bx(x-2) + x$$

3a  $x = 1$ ,  $-3 = -2 + B \cdot (-1) + 1$ ,  $-2 = B \cdot (-1)$ ,  $B = 2$

$$I = -2 \int \frac{dx}{x} + 2 \int \frac{dx}{x-2} + \int \frac{dx}{(x-2)^2} = -2 \ln|x| + 2 \ln|x-2| - \frac{1}{x-2} + C$$

$$I = \ln \left( \frac{x-2}{x} \right)^2 - \frac{1}{x-2} + C$$


---

2.5.

$$\frac{1}{x^3(x+1)^3} = \frac{A}{x^3} + \frac{B}{x^2} + \frac{C}{x} + \frac{D}{(x+1)^3} + \frac{E}{(x+1)^2} + \frac{F}{x+1}$$

$$A(x+1)^3 + Bx(x+1)^2 + Cx^2(x+1)^3 + Dx^3 + Ex^3(x+1) + Fx^3(x+1)^2 = 1.$$

За  $x = 0$  добијамо  $A = 1$ , а за  $x = -1$  добијамо  $D = -1$

$$Bx(x+1)^3 + Cx^2(x+1)^3 + Ex^3(x+1) + Fx^3(x+1)^2 = -3x(x+1)$$

$$B(x+1)^2 + Cx(x+1)^2 + Ex^2 + Fx^2(x+1) = -3$$

За  $x = 0$  добијамо  $B = -3$ , а за  $x = -1$  добијамо  $E = -3$

$$Cx(x+1)^2 + Fx^2(x+1) = 6x(x+1)$$

$$C(x+1) + Fx = 6$$

за  $x = 0$  добијамо  $C = 6$ , а за  $x = -1$  добијамо  $F = -6$

$$I = -\frac{1}{2x^2} + \frac{3}{x} + 6 \ln x + \frac{1}{2(x+1)^3} + \frac{3}{x+1} - 6 \ln(x+1) + C$$

2.6.

$$I = \frac{3}{2} \int \frac{2x+4-4-4/3}{x^2+4x+13} dx$$

$$I = \frac{3}{2} \int \frac{2x+4}{x^2+4x+13} dx - 8 \int \frac{d(x+2)}{(x+2)^2+3^2}$$

$$I = \frac{3}{2} \ln|x^2+4x+13| - \frac{8}{3} \arctan \frac{x+2}{3} + C$$

2.7.

$$\frac{x^5-1}{x^3+x^2+x} = x^2-x + \frac{x^2-1}{x^3+x^2+x}$$

$$I = \frac{x^3}{3} - \frac{x^2}{2} + J, \quad J = \int \frac{x^2-1}{x^3+x^2+x} dx$$

$$\frac{x^2-1}{x(x^2+x+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+x+1}$$

$$x^2-1 = A(x^2+x+1) + (Bx+C)x = (A+B)x^2 + (A+C)x + A$$

$A = 1, A+B=1, A+C=0, B=2, C=1$

$$\frac{x^2-1}{x^3+x^2+x} = -\frac{1}{x} + \frac{2x+1}{x^2+x+1}$$

$$J = -\ln|x| + \ln(x^2+x+1) + C = \ln \left| \frac{x^2+x+1}{x} \right| + C$$

2.8.  $\frac{x}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$

За  $x = 1, 1 = 3A, A = 1/3; \quad$  За  $x = 0, 0 = A + C \cdot (-1), C = 1/3$

За  $x = -1, -1 = A \cdot 1 + (-B + C) \cdot (-2), -1 = -1/3 + 2B, B = -1/3$

$$\begin{aligned}
I &= \frac{1}{3} \int \frac{dx}{x-1} - \frac{1}{3} \int \frac{x-1}{x^2+x+1} dx = \frac{1}{3} \ln|x-1| - \frac{1}{3} \cdot \frac{1}{2} \int \frac{2x-2}{x^2+x+1} dx \\
I &= \frac{1}{3} \ln|x-1| - \frac{1}{6} \int \frac{d(x^2+x+1)}{x^2+x+1} - \frac{1}{2} \int \frac{dx}{(x+1/2)^2 + (\sqrt{3}/2)^2} \\
I &= \frac{1}{3} \ln|x-1| - \frac{1}{6} \ln(x^2+x+1) - \frac{1}{2} \cdot \frac{2}{\sqrt{3}} \arctan \frac{x+1/2}{\sqrt{3}/2} + C
\end{aligned}$$


---

**2.9.**  $\frac{1}{x(1+x^2)^2} = \frac{A}{x} + \frac{Bx+C}{1+x^2} + \frac{Dx+E}{(1+x^2)^2}$

$$1 = A(1+x^2)^2 + (Bx+C)x(1+x^2) + (Dx+E)x$$

За  $x=0$  добијамо  $A=1$

За  $x=i$ ,  $1 = (Di+E)i$ ,  $1 = -D + Ei$ ,  $D = -1$ ,  $E = 0$

$$1 = (1+x^2)^2 + (Bx+C)x(1+x^2) - x^2$$

За  $x=1$ ,  $1 = 4 + (B+C) \cdot 2 - 1$ ,  $B+C = -1$

За  $x=-1$ ,  $1 = 4 + (-B+C) \cdot (-1) \cdot 2 - 1$

$$-2 = (B-C) \cdot 2, B-C = -1, 2B = -2, B = -1, C = 0$$

$$I \int \frac{dx}{x} - \int \frac{xdx}{1+x^2} - \int \frac{xdx}{(1+x^2)^2} = \ln|x| - \frac{1}{2} \ln(1+x^2) - \frac{1}{2} \int \frac{d(1+x^2)}{(1+x^2)^2}$$

$$I = \ln \frac{|x|}{\sqrt{1+x^2}} + \frac{1}{2(1+x^2)} + C$$


---

**3.1.**  $\tan \frac{x}{2} = t$ ,  $dx = \frac{2dt}{1+t^2}$ ,  $\sin x = \frac{2t}{1+t^2}$ ,  $\cos x = \frac{1-t^2}{1+t^2}$

$$I = \int \frac{1}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2} = \int \frac{2dt}{2+2t} = \int \frac{dt}{1+t}$$

$$I = \ln|1+t| + C = \ln\left(1 + \tan \frac{x}{2}\right) + C$$


---

**3.2.**  $f(x) = \frac{1 - \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}}{1 + \frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2}} = \frac{1+t^2-2t+1-t^2}{1+t^2+2t-1+t^2} = \frac{1-t}{t(1+t)}$

$$I = 2 \int \frac{(1-t)dt}{t(t+1)(t^2+1)}$$

$$\frac{1-t}{t(t+1)(t^2+1)} = \frac{A}{t} + \frac{B}{t+1} + \frac{Ct+D}{t^2+1}, \quad A=1, C=0, B=D=-1$$

$$I = 2 \int \frac{dt}{t} - 2 \int \frac{dt}{t+1} - 2 \int \frac{dt}{t^2+1} = 2 \ln \frac{t}{t+1} - 2 \arctan t + C$$

$$I = \ln \frac{\tan \frac{x}{2}}{\tan \frac{x}{2} + 1} - x + C$$


---

**3.3.**  $\tan x = t$

$$I = \int \frac{dx}{\cos^2 x + 9 \sin^2 x} = \int \frac{dt}{1 + (3t)^2} = \frac{1}{3} \arctan(3 \tan x) + C$$

---

**3.4.**  $\tan x = t$ 

$$I = \int \frac{1-t}{(1+t)(1+t^2)} dt$$

Међутим, може и без смене,

$$\begin{aligned} I &= \int \frac{\cos x - \sin x}{\sin x + \cos x} dx \\ &= \int \frac{d(\sin x + \cos x)}{\sin x + \cos x} \\ &= \ln |\sin x + \cos x| + C. \end{aligned}$$

---

**3.5.**  $\tan x = t$ ,  $\tan^2 t = t^2$ ,  $\frac{1 - \cos^2 t}{\cos^2 t} = t^2$ ,  $\frac{1}{\cos^2 t} = 1 + t^2$ 

$$I = \int \frac{dx/\cos^2 x}{1/\cos^2 x + \tan^2 x} = \int \frac{dt}{t^2 + 1 + t^2} = \int \frac{dt}{1 + 2t^2}$$

$$I = \frac{1}{\sqrt{2}} \int \frac{d(\sqrt{2}t)}{1 + (\sqrt{2}t)^2} = \frac{1}{\sqrt{2}} \cdot \arctan(\sqrt{2}t) + C$$

$$I = \frac{1}{\sqrt{2}} \arctan(\sqrt{2} \tan x) + C$$

Може и  $\frac{1}{1 + \sin^2 x} = \frac{1}{\cos^2 x + 2\sin^2 x} = \frac{1}{\cos^2 x} \cdot \frac{1}{1 + 2\tan^2 x}$

---

**4.1.**  $\sqrt{1-x} = t$ ,  $x = 1 - t^2$ ,  $dx = -2tdt$ ,  $x + 1 = -t^2 + 2$ 

$$I = 2 \int \frac{dt}{t^2 - 2} = \frac{1}{\sqrt{2}} \ln \left| \frac{t - \sqrt{2}}{t + \sqrt{2}} \right| + C = \frac{1}{\sqrt{2}} \ln \left| \frac{\sqrt{1-x} - \sqrt{2}}{\sqrt{1-x} + \sqrt{2}} \right| + C$$

---

**4.2.**  $\sqrt[3]{\frac{1-x}{1+x}} = t$ 

$$x = \frac{1-t^3}{1+t^3}, \quad dx = -\frac{6t^2}{(1+t^3)^2}, \quad 1+x = \frac{2}{1+t^3}$$

$$I = \frac{3}{2} \int t^3 dt = \frac{3}{8} t^4 + C = \frac{3}{8} \left( \frac{1-x}{1+x} \right)^{4/3} + C$$

---

**4.3.**  $\sqrt{\frac{x+1}{x-1}} = t$ ,  $\frac{x+1}{x-1} = t^2$ ,  $x = \frac{1+t^2}{t^2-1}$ ,  $dx = -\frac{4tdt}{(t^2-1)^2}$ 

$$I = \int \frac{\sqrt{\frac{x+1}{x-1}} - 1}{\sqrt{\frac{x+1}{x-1}} + 1} dx = -4 \int \frac{tdt}{(t-1)(t+1)^3}$$

$$\frac{t}{(t-1)(t+1)^3} = \frac{1}{8} \cdot \frac{1}{t-1} - \frac{1}{8} \cdot \frac{1}{t+1} - \frac{1}{4} \cdot \frac{1}{(t+1)^2} + \frac{1}{2} \cdot \frac{1}{(t+1)^3}$$



$$I = -\frac{1}{2} \ln|t-1| + \frac{1}{2} \ln|t+1| - \frac{1}{t+1} + \frac{1}{(t+1)^2} + C$$

$$I = \frac{1}{2} \ln \left| \frac{t+1}{t-1} \right| - \frac{t}{(t+1)^2} + C, \quad t = \sqrt{\frac{x+1}{x-1}}$$


---

4.4.  $(1-x)\sqrt{1-x^2} = (1-x)\sqrt{(1-x)(1+x)} = (1-x)(1+x)\sqrt{\frac{1-x}{1+x}}$

$$\frac{1-x}{1+x} = t^2, \quad x = \frac{1-t^2}{1+t^2}, \quad dx = \frac{-4tdt}{(1+t^2)^2}, \quad 1-x = \frac{2t^2}{1+t^2}, \quad 1+x = \frac{2}{1+t^2}$$

$$I = - \int \frac{4tdt}{(1+t^2)^2} \cdot \frac{2t^2}{1+t^2} \cdot \frac{2}{1+t^2} \cdot t = - \int \frac{dt}{t^2} = \frac{1}{t} + C$$

$$I = \sqrt{\frac{1+x}{1-x}} + C$$

Друго решење.

$$x = \sin t$$

$$\begin{aligned} I &= \int \frac{dt}{1-\sin t} = \int \frac{1+\sin t}{1-\sin^2 t} dt \\ &= \int \frac{1+\sin t}{\cos^2 t} dt = \tan t - \int \frac{d(\cos t)}{\cos^2 t} \\ &= \tan t + \frac{1}{\cos t} + C = \frac{\sin t + 1}{\cos t} + C \\ &= \frac{\sin t + 1}{\sqrt{1-\sin^2 t}} = \frac{x+1}{\sqrt{1-x^2}} + C \\ &= \sqrt{\frac{1+x}{1-x}} + C \end{aligned}$$


---

4.5.  $x = \sin t$

$$\begin{aligned} I &= \int \frac{\cos t dt}{\cos t - 1} \\ &= \int dt - \int \frac{dt}{1-\cos t} \\ &= t - \int \frac{1+\cos t}{\sin^2 t} dt \\ &= t + \cot t + \frac{1}{\sin t} + C \\ &= \arcsin x + \frac{\sqrt{1-x^2}}{x} + \frac{1}{x} + C. \end{aligned}$$


---

4.6.  $x = \cosh t$

$$\begin{aligned}
I &= \int \sqrt{(\cosh^2 t - 1)^3} \sinh t dt \\
&= \int \sinh^4 t dt \\
&= \frac{1}{4} \int (\cosh 2t - 1)^2 dt \\
&= \frac{1}{4} \int \cosh^2 t dt - \frac{1}{2} \int \cosh 2t dt + \frac{1}{4} \int dt \\
&= \frac{1}{32} \sinh 4t - \frac{1}{4} \sinh 2t + \frac{3}{8} t + C.
\end{aligned}$$

Како је

$$\begin{aligned}
t &= \operatorname{arcosh} x = \ln(x + \sqrt{x^2 - 1}), \\
\sinh 2t &= 2 \sinh t \cosh t = 2x\sqrt{x^2 - 1}, \\
\sinh 4t &= 2 \sinh 2t \cosh 2t = 4x\sqrt{x^2 - 1}(2x^2 - 1),
\end{aligned}$$

имамо да је

$$\begin{aligned}
I &= \frac{1}{8} x(2x^2 - 1)\sqrt{x^2 - 1} - \frac{1}{2} x\sqrt{x^2 - 1} + \\
&\quad \frac{3}{8} \ln(x + \sqrt{x^2 - 1}) + C.
\end{aligned}$$

#### 4.7. $x = \sinh t$

$$\begin{aligned}
I &= \int \frac{\cosh t dt}{\sinh^2 t \cosh t} \\
&= \int \frac{dt}{\sinh^2 t} \\
&= -\operatorname{coth} t + C \\
&= -\frac{\sqrt{1 + \sinh^2 t}}{\sinh t} + C \\
&= -\frac{\sqrt{1 + x^2}}{x} + C.
\end{aligned}$$

#### 4.8. $x = \tan t$

$$I = \int \frac{\cos t}{\sin^4 t} dt = -\frac{1}{3 \sin^3 t} + C$$

Како је

$$\sin^2 t = \frac{\sin^2 t}{\sin^2 t + \cos^2 t} = \frac{\tan^2 t}{1 + \tan^2 t} = \frac{x^2}{1 + x^2},$$

то је  $\sin t = \frac{x}{\sqrt{1 + x^2}}$ , па је

$$I = -\frac{1}{3} \frac{\sqrt{(1 + x^2)^3}}{x^3} + C$$

#### 4.9. $I = \int \sqrt{(x + 3)^2 - 3^2} d(x + 3)$

$$\int \sqrt{t^2 - a^2} dt = \frac{t}{2} \sqrt{t^2 - a^2} - \frac{a^2}{2} \ln |t + \sqrt{t^2 - a^2}| + C$$

$$I = \frac{x+3}{2} \sqrt{x^2 + 6x} - \frac{9}{2} \ln |x+3 + \sqrt{x^2 + 6x}| + C$$


---

4.10.

$$\begin{aligned} I &= \frac{5}{2} \int \frac{2x + 8/5}{\sqrt{x^2 + 2x + 5}} dx \\ &= \frac{5}{2} \int \frac{2x + 2}{\sqrt{x^2 + 2x + 5}} dx - \frac{5}{2} \cdot \frac{2}{5} \int \frac{dx}{\sqrt{x^2 + 2x + 5}} \\ &= 5 \int \frac{d(x^2 + 2x + 5)}{2\sqrt{x^2 + 2x + 5}} dx - \int \frac{dx}{\sqrt{(x+1)^2 + 2^2}} \\ &= 5\sqrt{x^2 + 2x + 5} - \ln(x+1 + \sqrt{x^2 + 2x + 5}) + C \end{aligned}$$


---

4.11.

$$\begin{aligned} I &= \frac{1}{2} \int \frac{2x+1}{\sqrt{x^2+x+1}} dx - \frac{1}{2} \int \frac{dx}{\sqrt{x^2+x+1}} \\ &= \sqrt{x^2+x+1} - \frac{1}{2} \ln \left( x + \frac{1}{2} + \sqrt{x^2+x+1} \right). \end{aligned}$$


---

4.12.

$$\begin{aligned} I &= \int \frac{x^2+x+1}{\sqrt{x^2+x+1}} dx - \int \frac{x+1/2}{\sqrt{x^2+x+1}} dx - \frac{1}{2} \int \frac{dx}{\sqrt{x^2+x+1}} \\ &= \int \sqrt{(x+1/2)^2 + 3/4} dx - \int \frac{2x+1}{2\sqrt{x^2+x+1}} dx - \frac{1}{2} \int \frac{dx}{\sqrt{(x+1/2)^2 + 3/4}} \\ &= \frac{x+1/2}{2} \sqrt{x^2+x+1} + \frac{3}{8} \ln(x+1/2 + \sqrt{x^2+x+1}) \\ &\quad - \sqrt{x^2+x+1} - \frac{1}{2} \ln(x+1/2 + \sqrt{x^2+x+1}) + C \\ &= \left( \frac{2x+1}{4} - 1 \right) \sqrt{x^2+x+1} + \left( \frac{3}{8} - \frac{1}{2} \right) \ln(x+1/2 + \sqrt{x^2+x+1}) + C \\ &= \frac{2x-3}{4} \sqrt{x^2+x+1} - \frac{1}{8} \ln(x+1/2 + \sqrt{x^2+x+1}) + C \end{aligned}$$


---

4.13.  $I = \int \sqrt{(2x-1)^2 + 2} dx, \quad 2x-1 = t, \quad 2dx = dt$

$$\begin{aligned} I &= \frac{1}{2} \int \sqrt{t^2 + 2} dt \\ &= \frac{1}{2} \left( \frac{t}{2} \sqrt{t^2 + 2} + \frac{2}{2} \ln(t + \sqrt{t^2 + 2}) \right) + C \\ &= \frac{2x-1}{4} \sqrt{4x^2 - 4x + 3} + \frac{1}{2} \ln(2x-1 + \sqrt{4x^2 - 4x + 3}) + C \end{aligned}$$


---

4.14.  $x = 2 \sin t$

$$I = \int 4 \sin^2 t \cdot 2 \cos t \cdot 2 \cos t dt = 16 \int \sin^2 t \cos^2 t dt = 16J$$

$$\begin{aligned} \sin^2 t \cos^2 t &= \frac{1 - \cos 2t}{2} \cdot \frac{1 + \cos 2t}{2} \\ &= \frac{1}{4}(1 - \cos^2 2t) = \frac{1}{4} \left(1 - \frac{1 + \cos 4t}{2}\right) = \frac{1}{8} - \frac{1}{8} \cos 4t \end{aligned}$$

$$I = 16J = 16 \left( \frac{1}{8}t - \frac{1}{32} \sin 4t \right) + C = 2t - \frac{1}{2} \sin 4t + C$$

Како је

$$\begin{aligned} \sin 4t &= 2 \sin 2t \cos 2t \\ &= 4 \sin t \cdot \cos t \cdot (\cos^2 t - \sin^2 t) \\ &= 4 \sin t \sqrt{1 - \sin^2 t} (1 - 2 \sin^2 t) \\ &= 4 \cdot \frac{x}{2} \sqrt{1 - \frac{x^2}{4}} \left(1 - 2 \cdot \frac{x^2}{4}\right) \\ &= x \sqrt{4 - x^2} \left(1 - \frac{x^2}{2}\right) \\ &= \frac{x}{2} \sqrt{4 - x^2} (2 - x^2) \end{aligned}$$

то је

$$I = 2 \arcsin \frac{x}{2} - \frac{x}{4} \sqrt{4 - x^2} (2 - x^2) + C$$

Друго решење.

$$u = x, \quad dv = x \sqrt{4 - x^2} dx, \quad du = dx, \quad v = -\frac{1}{3} (4 - x^2) \sqrt{4 - x^2}$$

$$I = -\frac{x}{3} (4 - x^2)^{3/2} + \frac{1}{3} (4 - x^2) \sqrt{4 - x^2} dx$$

$$I = -\frac{x}{3} (4 - x^2)^{3/2} + \frac{4}{3} \int \sqrt{4 - x^2} - \frac{1}{3} I$$

$$\frac{4}{3} I = -\frac{x}{3} (4 - x^2)^{3/2} + \frac{4}{3} J, \quad J = \int \sqrt{4 - x^2} dx$$

$$J = \int 2^2 - x^2 dx = 2 \arcsin \frac{x}{2} + \frac{x}{2} \sqrt{4 - x^2} + C$$

$$I = -x(4 - x^2)^{3/2} + 8 \arcsin \frac{x}{2} + 2x \sqrt{4 - x^2} + C$$

$$I = 2 \arcsin \frac{x}{2} + x \sqrt{4 - x^2} \left( \frac{1}{2} - 1 + \frac{x^2}{4} \right) + C \quad I = 2 \arcsin \frac{x}{2} + \frac{x}{4} \sqrt{4 - x^2} (x^2 - 2) + C$$

**4.15.** Ојлерова смена  $\sqrt{1 + x - x^2} = tx - 1$

$$1 + x - x^2 = t^2 x^2 - 2tx + 1, \quad x(1 - x) = x(t^2 x - 2t), \quad 1 - x = t^2 x - 2t$$

$$x = \frac{1 + 2t}{t^2 + 1}, \quad dx = -2 \cdot \frac{t^2 + t - 1}{(t^2 + 1)^2} dt$$

$$\sqrt{1 + x - x^2} = tx - 1 = t \cdot \frac{1 + 2t}{t^2 + 1} = \frac{t^2 + t - 1}{t^2 + 1}$$

$$1 + x = \frac{t^2 + 2t + 2}{t^2 + 1}$$

$$I = -2 \int \frac{t^2 + 1}{t^2 + 2t + 2} \cdot \frac{t^2 + 1}{t^2 + t - 1} \cdot \frac{t^2 + t - 1}{(t^2 + 1)^2} dt = -2 \int \frac{dt}{1 + (t + 1)^2}$$

$$I = -2 \arctan(t+1) + C = -2 \arctan \frac{1+x+\sqrt{1+x-x^2}}{x} + C$$


---

$$5.1. \int \sqrt{a^2 - x^2} dx = \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2} + C$$

$$F(x) = \frac{2^2}{2} \arcsin \frac{x}{2} + \frac{x}{2} \sqrt{4 - x^2}$$

$$I = F(x) \Big|_0^1 = 2 \arcsin \frac{1}{2} + \frac{1}{2} \sqrt{4 - 1} = 2 \cdot \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$

$$I = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$


---

$$5.2. I = \int_0^{\pi/4} \frac{dx}{\cos^2 x + 3 \sin^2 x} = \int_0^{\pi/4} \frac{d(\tan x)}{1 + 3 \tan^2 x}$$

$$I = \frac{1}{\sqrt{3}} \int_0^{\pi/4} \frac{d(\sqrt{3} \tan x)}{1 + (\sqrt{3} \tan x)^2}$$

$$I = \frac{1}{\sqrt{3}} \arctan(\sqrt{3} \tan x) \Big|_0^{\pi/4}$$

$$I = \frac{1}{\sqrt{3}} \arctan \sqrt{3} = \frac{\pi}{3\sqrt{3}}$$


---

$$5.3. I = - \int_{1/e}^1 \ln x dx + \int_1^e \ln x dx$$

$$\int \ln x dx = x \ln x - x + C$$

$$I = -(x \ln x - x) \Big|_{1/e}^1 + x \ln x - x \Big|_1^e$$

$$I = - \left( 0 - 1 - \frac{1}{e} \ln \frac{1}{e} + \frac{1}{e} \right) + e - e - (0 - 1)$$

$$I = 1 - \frac{2}{e} + 1 = 2 - \frac{2}{e}$$


---

5.4.

$$F(x) = \frac{2}{\sqrt{5}} \arctan \left( \frac{1}{\sqrt{5}} \tan \frac{x}{2} \right), \quad x \in [0, \pi)$$

$$I = F(x) \Big|_0^{\pi-} = F(\pi-) - F(0)$$

$$I = \lim_{x \rightarrow \pi-} \frac{2}{\sqrt{5}} \arctan \left( \frac{1}{\sqrt{5}} \tan \frac{x}{2} \right) - 0$$

$$I = \frac{\pi}{\sqrt{5}}$$


---

$$5.5. \quad x = a \sin t, \quad dx = a \cos t dt, \quad \begin{array}{c|c|c} x & 0 & a \\ \hline t & 0 & \pi/2 \end{array}$$

$$I = \int_0^{\pi/2} a^2 \sin^2 t \sqrt{a^2 - a^2 \sin^2 t} a \cos t dt = a^4 \int_0^{\pi/2} \sin^2 t \cos^2 t dt$$

$$I = \frac{a^4}{4} \int_0^{\pi/2} \sin^2 2t dt = \frac{a^4}{8} \int_0^{\pi/2} (1 - \cos 4t) dt$$

$$I = \frac{a^4}{8} \left( t - \frac{1}{4} \sin 4t \right) \Big|_0^{\pi/2}$$

$$I = \frac{\pi}{16} a^4$$


---

$$5.6. \quad x = 2 \sin t, \quad dx = 2 \cos t dt, \quad \frac{x}{t} \Big|_{\pi/6}^{\sqrt{3}} \Big|_{\pi/3}$$

$$\int_{\pi/6}^{\pi/3} \frac{(8 \sin^3 t + 1) 2 \cos t}{4 \sin^2 t \sqrt{4 - 4 \sin^2 t}} dt$$

$$I = 2 \int_{\pi/6}^{\pi/3} \sin t dt + \frac{1}{4} \int_{\pi/6}^{\pi/3} \frac{dt}{\sin^2 t}$$

$$I = -2 \cos t \Big|_{\pi/6}^{\pi/3} - \frac{1}{4} \cos t \Big|_{\pi/6}^{\pi/3} = -2 \left( \frac{1}{2} - \frac{\sqrt{3}}{2} \right) - \frac{1}{4} \left( \frac{\sqrt{3}}{3} - \sqrt{3} \right)$$

$$I = \frac{7}{6} \sqrt{3} - 1$$


---

$$5.7. \quad x^4 = t, \quad x^3 dx = \frac{1}{4} dt$$

$$I = \frac{1}{4} \int_0^4 \sqrt{16 - t^2} dt$$

$$t = 4 \sin u$$

$$I = 4 \int_0^{\pi/2} |\cos u| \cdot \cos u u = 4 \int_0^{\pi/2} \cos^2 u du$$

$$I = 2 \int_0^{\pi/2} (1 + \cos 2u) du = 2u + \sin 2u \Big|_0^{\pi/2}$$

$$I = \pi$$


---

$$5.8. \quad I = \int_4^5 x \sqrt{(x-2)^2 - 4} dx$$

$$x - 2 = t, \quad dx = dt$$

$$I = \int_2^3 (2+t) \sqrt{t^2 - 4} dt$$

$$I = 2 \int_2^3 \sqrt{t^2 - 4} dt + \int_2^3 t \sqrt{t^2 - 4} dt$$

$$I = 2 \left( \frac{1}{2} \sqrt{t^2 - 4} - 2 \ln |t + \sqrt{t^2 - 4}| \right) \Big|_2^3 + \frac{1}{3} (t^2 - 4)^{3/2} \Big|_2^3$$

$$I = \frac{14}{3} \sqrt{5} - 4 \ln \frac{3 + \sqrt{5}}{2}$$


---

$$5.9. \quad \tan \frac{x}{2} = t, \quad \frac{x}{t} \Big|_{\frac{0}{0}} \Big|_{\frac{\pi/2}{1}}, \quad \sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$$

$$I = \int_0^1 \frac{1}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2}$$

$$I = \int_0^1 \frac{2dt}{1+t^2+2t+1-t^2} = \int_0^1 \frac{2dt}{2+2t} = \int_0^1 \frac{dt}{1+t}$$

$$I = \ln(1+t) \Big|_0^1 = \ln 2$$


---

**5.10.**  $x = \frac{\pi}{2} - t, \quad dx = -dt, \quad \begin{array}{c|c|c} x & 0 & \pi/2 \\ \hline t & \pi/2 & 0 \end{array}$

$$I = \int_{\pi/2}^0 \frac{\cos^3 t}{\sin^3 t + \cos^3 t} (-dt) = \int_0^{\pi/2} \frac{\cos^3 t}{\sin^3 t + \cos^3 t} dt$$

$$2I = \int_0^{\pi/2} \frac{\sin^3 x + \cos^3 x}{\sin^3 x + \cos^3 x} dx = \int_0^{\pi/2} dt = \frac{\pi}{2}$$

$$I = \pi/4$$


---

**5.11.**  $x = \pi - t, \quad dx = -dt, \quad \begin{array}{c|c|c} x & 0 & \pi \\ \hline t & \pi & 0 \end{array}$

$$I = \pi \int_0^\pi \frac{\sin x dx}{1 + \cos^2 x} - \int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx$$

$$I = \pi J - I, \quad 2I = \pi J, \quad I = \frac{\pi}{2} J$$

$$J = - \int_0^\pi \frac{d(\cos x)}{1 + \cos^2 x} = - \arctan(\cos x) \Big|_0^\pi = -(\arctan(-1) - \arctan 1) = \frac{\pi}{2}$$

$$I = \frac{\pi^2}{4}$$


---

**5.12.**  $\cos x = t$

$$I = \int_{\pi/4}^{\pi/3} \frac{\sin x dx}{(1 - \cos^2 x) \cos^3 x} = \int_{1/2}^{\sqrt{2}/2} \frac{dt}{(1-t^2)t^3}$$

$$\frac{1}{(1-t^2)t^3} = \frac{1}{2} \cdot \frac{1}{1-t} - \frac{1}{2} \cdot \frac{1}{1+t} + \frac{1}{t} + \frac{1}{t^3}$$

$$I = \frac{1}{2} \int_{1/2}^{\sqrt{2}/2} \frac{dt}{1-t} - \frac{1}{2} \int_{1/2}^{\sqrt{2}/2} \frac{dt}{1+t} + \int_{1/2}^{\sqrt{2}/2} \frac{dt}{t} + \int_{1/2}^{\sqrt{2}/2} \frac{dt}{t^3}$$

$$I = -\frac{1}{2} \ln(t^2 - 1) + \ln t - \frac{1}{2} \cdot \frac{1}{t^2} \Big|_{1/2}^{\sqrt{2}/2}$$

$$I = \frac{1}{2} \ln 3 + 1$$


---

**5.13.**  $u = \ln \frac{1+x}{1-x}, \quad dv = x dx \quad du = \frac{2dx}{1-x^2}, \quad v = \frac{x^2}{2}$

$$I = \frac{x^2}{2} \ln \frac{1+x}{1-x} \Big|_1^{1/2} - \int_0^{1/2} \frac{x^2 dx}{1-x^2}$$

$$I = \frac{1}{8} \ln 3 - \int_0^{1/2} \frac{x^2 - 1 + 1}{1-x^2} dx$$

$$I = \frac{1}{8} \ln 3 + x \Big|_0^{1/2} - \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right| \Big|_0^{1/2}$$

$$I = \frac{1}{2} - \frac{3}{8} \ln 3$$

---

5.14.  $u = \ln^2 x dx$ ,  $dv = x^2 dx$ ,  $du = \frac{2 \ln x}{x} dx$ ,  $v = \frac{x^3}{3}$

$$I = \frac{x^3}{3} \ln^2 x \Big|_0^e - \frac{2}{3} \int_0^e x^2 \ln x dx = \frac{e^3}{3} - \frac{2}{3} J$$

3a  $J$   $u = \ln x$ ,  $dv = x^2 dx$ ,  $du = \frac{dx}{x}$ ,  $v = \frac{x^3}{3}$

$$J = \frac{x^3}{3} \ln x \Big|_0^e - \frac{1}{3} \int_0^e x^2 dx = \frac{e^3}{3} - \frac{1}{3} \cdot \frac{x^3}{3} \Big|_0^e = \frac{2}{9} e^3$$

$$I = \frac{e^3}{3} - \frac{4}{27} e^3 = \frac{5}{27} e^3$$


---

5.15.  $u = \arcsin \sqrt{\frac{x}{x+1}}$ ,  $dv = dx$   $du = \frac{1}{\sqrt{1-\frac{x}{x+1}}} \cdot \frac{1}{2\sqrt{\frac{x}{x+1}}} \cdot \frac{x+1-x}{(x+1)^2} dx = \frac{1}{2\sqrt{x}} \cdot \frac{dx}{x+1}$ ,  $v = x$

$$I = x \arcsin \sqrt{\frac{x}{x+1}} \Big|_0^3 - \int_0^3 \frac{x dx}{2\sqrt{x}(x+1)} = 3 \arcsin \sqrt{\frac{3}{4}} - \int_0^3 \frac{\sqrt{x^2} d(\sqrt{x})}{1+\sqrt{x^2}}$$

$$I = 3 \arcsin \frac{\sqrt{3}}{2} - (\sqrt{x} - \arctan \sqrt{x}) \Big|_0^3$$

$$I = 3 \cdot \frac{\pi}{3} - \sqrt{3} + \frac{\pi}{3} = \frac{4}{3} \pi - \sqrt{3}$$


---

5.16.  $u = \ln(x + \sqrt{1+x^2})$ ,  $dv = \frac{xdx}{(1+x^2)^2}$ ,  $du = \frac{dx}{\sqrt{1+x^2}}$ ,  $v = \frac{-1}{2(1+x^2)}$

$$I = -\frac{\ln(x + \sqrt{1+x^2})}{2(1+x^2)} \Big|_0^1 + \frac{1}{2} \int_0^1 (1+x^2)^{-3/2} dx = -\frac{1}{4} \ln(1+\sqrt{2}) + \frac{1}{2} J$$

3a  $J$ :  $x = \tan t$ ,  $t \in [0, \pi/4]$

$$J = \int_0^{\pi/4} (1 + \tan^2 t)^{-3/2} \cos^{-2} t dt = \int_0^{\pi/4} \cos t dt = \frac{\sqrt{2}}{4}$$

$$I = \frac{\sqrt{2}}{4} - \frac{1}{4} \ln(1 + \sqrt{2})$$


---

5.17.  $I_n = \int_0^a (a^2 - x^2)^{n-1} (a^2 - x^2) dx = a^2 I_{n-1} - J$

3a  $J$ :  $u = x$ ,  $x(a^2 - x^2)^{n-1} dx = dv$   $du = dx$ ,  $v = -\frac{1}{2n}(a^2 - x^2)^n$

$$J = -\frac{x}{2n} (a^2 - x^2)^n \Big|_0^a + \frac{1}{2n} \int_0^a (a^2 - x^2)^n dx = \frac{1}{2n} I_n$$

$$I_n = a^2 I_{n-1} - \frac{1}{2n} I_n, \quad I_n = \frac{2n}{2n+1} a^2 I_{n-1}$$

$$I_n = \frac{2n}{2n+1} a^2 I_{n-1} = \frac{2n}{2n+1} \cdot \frac{2(n-1)}{2(n-1)+1} a^4 I_{n-2} = a^{2n} \frac{(2n)!!}{(2n+1)!!} I_0$$

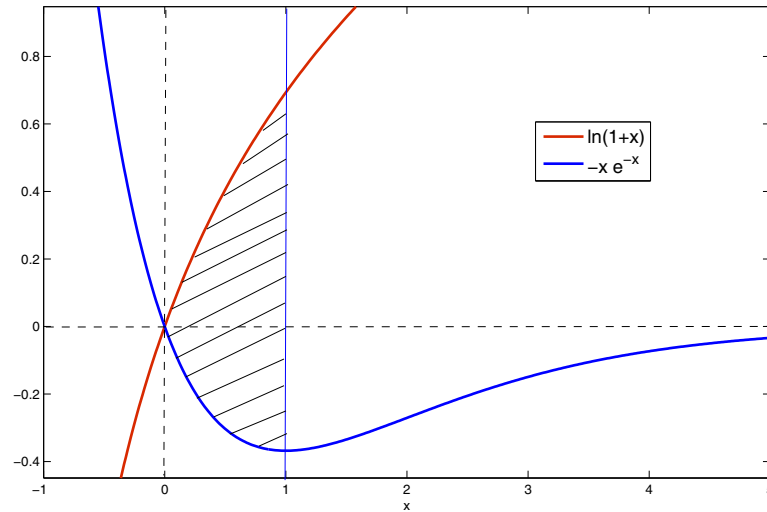
$$I_0 = \int_0^a dx = a, \quad I_n = a^{2n+1} \frac{(2n)!!}{(2n+1)!!}$$


---

6.1.  $P = \int_0^1 \ln(1+x) dx + \int_0^1 x e^{-x} dx$



$$P = (x+1)\ln(1+x) - x \Big|_0^1 + (-xe^{-x} - e^{-x}) \Big|_0^1$$



$$P = 2\ln 2 - \frac{2}{e} = \ln 4 - \frac{2}{e}$$

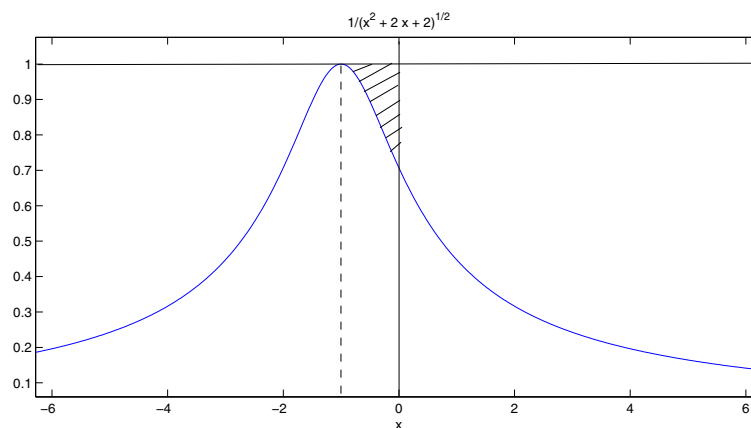
6.2.  $y = \frac{1}{\sqrt{x^2 + 2x + 2}}$ ,  $y = 1, x = 0$ ,  $P = ?$

Решение.  $P = \int_{-1}^0 \left(1 - \frac{1}{\sqrt{x^2 + 2x + 2}}\right) dx$

$$P = x \Big|_{-1}^0 - \int_{-1}^0 \frac{dx}{\sqrt{x^2 + 2x + 2}}$$

$$P = 1 - \int_{-1}^0 \frac{dx}{\sqrt{(x+1)^2 + 1}}$$

$$P = 1 - \ln(x+1 + \sqrt{x^2 + 2x + 2}) \Big|_{-1}^0$$



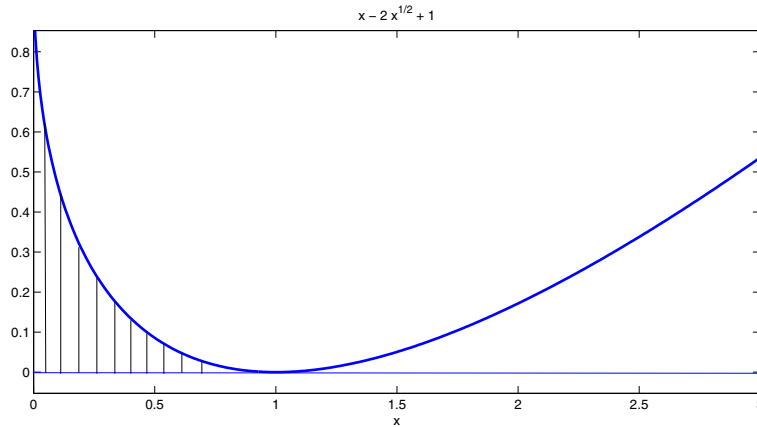
$$P = 1 - \ln(1 + \sqrt{2}) \approx 0.1186$$

6.3.  $y = 0, \quad x = a^2$

$x = 0, \quad y = a^2$

$y = (a - \sqrt{x})^2$

$$P = \int_0^{a^2} (a^2 + x - 2a\sqrt{x}) dx$$



$$P = \frac{a^4}{6}$$

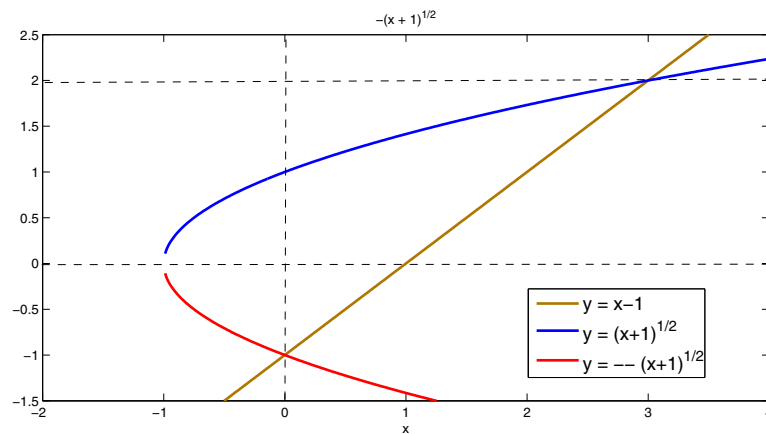
6.4.  $P_1 = \int_{-1}^0 2\sqrt{x+1} dx$

$$P_1 = \frac{4}{3}(x+1)^{3/2} \Big|_{-1}^0$$

$$P_2 = \int_0^3 (\sqrt{x+1} - x + 1) dx$$

$$P_2 = \frac{2}{3}(x+1)^{3/2} - \frac{(x-1)^2}{2} \Big|_0^3$$

$$P = P_1 + P_2 = \frac{9}{2}$$



Друго решење.

$$P = \int_{-1}^2 (y+1-y^2+1)dy = -\frac{y^3}{3} + \frac{y^2}{2} + 2y \Big|_{-1}^2$$


---

$$6.5. f'(x) = x - \frac{1}{4x}, \quad f'^2(x) = x^2 + \frac{1}{16x^2} - \frac{1}{2}$$

$$1 + f'^2(x) = \frac{1}{2} + x^2 + \frac{1}{16x^2} = \left(x + \frac{1}{4x}\right)^2$$

$$\sqrt{1 + f'^2(x)} = x + \frac{1}{4x}$$

$$l = \int_1^3 \left(x + \frac{1}{4x}\right) dx = \frac{x^2}{2} + \frac{1}{4} \ln x \Big|_1^3$$

$$l = \frac{9}{2} + \frac{1}{4} \ln 3 - \frac{1}{2} = 4 + \frac{1}{4} \ln 3$$


---

$$6.6. f'(x) = \frac{2x}{x^2-1}, \quad 1 + f'^2(x) = 1 + \frac{4x^2}{(x^2-1)^2}$$

$$1 + f'^2(x) = \frac{(x^2-1)^2 + 4x^2}{(x^2-1)^2} = \frac{(x^2+1)^2}{(x^2-1)^2}$$

$$l = \int_2^5 \frac{x^2+1}{x^2-1} dx = \int_2^5 dx + 2 \int_2^5 \frac{dx}{x^2-1} = x \Big|_2^5 + 2 \cdot \frac{1}{2} \ln \left| \frac{x-1}{x+1} \right| \Big|_2^5$$

$$l = 3 + \ln \frac{2}{3} - \ln \frac{1}{3} = 3 + \ln 2$$


---

$$6.7. y' = \cot x, \quad 1 + y'^2(x) = 1 + \cot^2 x = \frac{1}{\sin^2 x}$$

$$l = \int_{\pi/3}^{2\pi/3} \frac{dx}{\sin x} = \frac{1}{2} \int_{\pi/3}^{2\pi/3} \frac{dx}{\sin \frac{x}{2} \cos \frac{x}{2}}$$

$$l = \int_{\pi/3}^{2\pi/3} \frac{d(\tan \frac{x}{2})}{\tan \frac{x}{2}} = \ln \left( \tan \frac{x}{2} \right) \Big|_{\pi/3}^{2\pi/3}$$

$$l = \ln \sqrt{3} - \ln \frac{1}{\sqrt{3}} = \ln \sqrt{3} + \ln \sqrt{3} = \ln 3$$


---

$$6.8. y' = \frac{e^{2x}}{e^{2x}-1}, \quad 1 + y'^2 = \frac{(e^{2x}+1)^2}{(e^{2x}-1)^2}$$

$$l = \int_1^2 \frac{e^{2x}+1}{e^{2x}-1} dx = \int_e^{e^2} \frac{t^2+1}{t(t^2-1)} dt$$

$$l = \int_e^{e^2} \left( \frac{1}{t-1} + \frac{1}{t+1} - \frac{1}{t} \right) dt = \ln \frac{|t^2-1|}{|t|} \Big|_e^{e^2}$$

$$l = -1 + \ln(e^2+1)$$


---

$$6.9. y' = x$$

$$l = \int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{1+x^2} dx = 2 \int_0^{\sqrt{2}} \sqrt{1+x^2} dx = 2 \int_0^{\sqrt{2}} \frac{1+x^2}{\sqrt{1+x^2}} dx = 2 \int_0^{\sqrt{2}} \left( \frac{1}{\sqrt{1+x^2}} + \frac{x^2}{\sqrt{1+x^2}} \right) dx = 2 \ln(x + \sqrt{1+x^2}) \Big|_0^{\sqrt{2}} + 2J$$

$$\text{3a } J: u = x, \quad dv = \frac{x dx}{\sqrt{1+x^2}}, \quad v = \sqrt{1+x^2}$$

$$2J = 2x\sqrt{1+x^2} \Big|_0^{\sqrt{2}} - 2 \int_0^{\sqrt{2}} \sqrt{1+x^2} dx = 2\sqrt{6} - l$$

$$l = 2 \ln(\sqrt{2} + \sqrt{3}) + 2\sqrt{6} - l$$

$$l = \ln(\sqrt{2} + \sqrt{3}) + \sqrt{6}$$

$$\text{6.10. } y' = \frac{x + 4\sqrt{6}}{\sqrt{x^2 - 48}}, \quad \sqrt{1+y'^2} = \sqrt{2} \cdot \frac{x + 2\sqrt{6}}{\sqrt{x^2 - 48}}$$

$$l = \int_7^8 \frac{\sqrt{2}}{2} \cdot \frac{d(x^2 - 48)}{\sqrt{x^2 - 48}} + \sqrt{2} \cdot 2\sqrt{6} \int_7^8 \frac{dx}{\sqrt{x^2 - (4\sqrt{3})^2}}$$

$$l = \frac{\sqrt{2}}{2} \cdot 2\sqrt{x^2 - 48} \Big|_7^8 + 2\sqrt{12} \ln(x + \sqrt{x^2 - 48}) \Big|_7^8$$

$$l = 3\sqrt{2} + 4\sqrt{3} \ln \frac{3}{2}$$

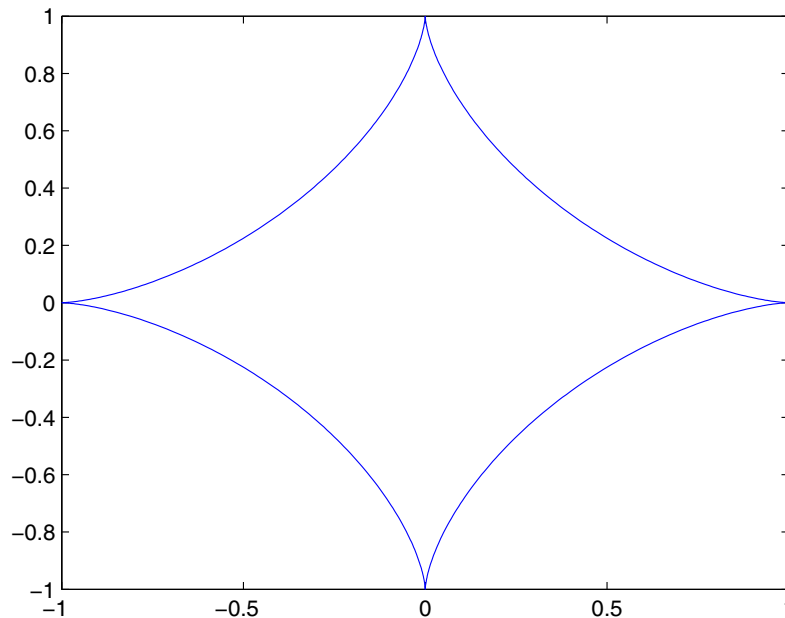
$$\text{6.11. } x'^2 = 9a^2 \cos^4 t \sin^2 t, \quad y'^2 = 9a^2 \sin^4 t \cos^2 t$$

$$x'^2 + y'^2 = 9a^2 \cos^2 t \sin^2 t (\cos^2 t + \sin^2 t)$$

$$x'^2 + y'^2 = 9a^2 \cos^2 t \sin^2 t$$

$$l = 4 \cdot 3a \int_0^{\pi/2} \cos t \sin t dt = 12a \int_0^{\pi/2} \sin t d(\sin t)$$

$$l = 6a \sin^2 t \Big|_0^{\pi/2} = 6a$$



Зашто не може:

$$l = 3a \int_0^{2\pi} \cos t \sin t dt = \frac{3a}{2} \sin^2 t \Big|_0^{2\pi} = 0?$$

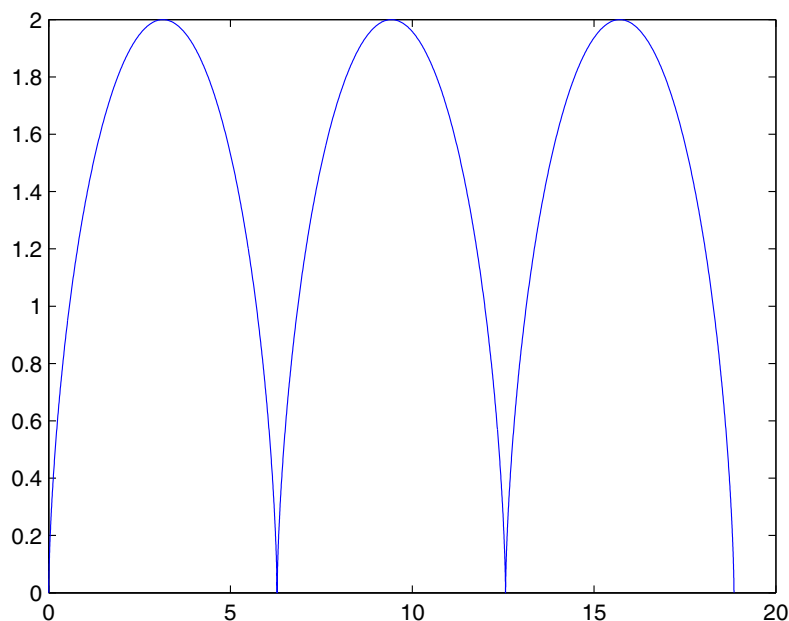
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**6.12.**  $x' = a(1 - \cos t)$ ,  $y' = a \sin t$

$$x'^2 + y'^2 = a^2(1 + \cos^2 t - 2 \cos t + \sin^2 t)$$

$$x'^2 + y'^2 = 2a^2(1 - \cos t) = 4a^2 \sin^2 \frac{t}{2}$$

$$l = \int_0^{2\pi} 2a \sin \frac{t}{2} dt = 2a \left( -\cos \frac{t}{2} \right) \cdot 2 \Big|_0^{2\pi}$$



$$l = 4a(1 + 1) = 8a$$


---

**6.13.**  $x'(t) = -2\sqrt{2} \frac{t}{\sqrt{1-t^2}}$ ,  $y'(t) = \sqrt{1-t^2} - \frac{t^2}{\sqrt{1-t^2}}$

$$x'^2 + y'^2 = \frac{(2t^2 + 1)^2}{1-t^2}, \quad \sqrt{x'^2 + y'^2} = \frac{2t^2 + 1}{\sqrt{1-t^2}}$$

$$l = \int_0^1 \frac{2t^2 + 1}{\sqrt{1-t^2}} dt = -2I + 3 \arcsin t \Big|_0^1 = -2I + \frac{3}{2}\pi$$

$$I = \int_0^1 \sqrt{1-t^2} dt = \int_0^{\pi/2} \cos^2 u du$$

$$I = \int_0^{\pi/2} \frac{1 + \cos 2u}{2} du = \frac{1}{2}u \Big|_0^{\pi/2} + \frac{1}{4} \sin 2u \Big|_0^{\pi/2} = \frac{\pi}{4}$$

$$l = -2 \cdot \frac{\pi}{4} + \frac{3}{2}\pi = \pi$$


---

**6.14.**  $V_x = \pi \int_{-a}^a y^2 dx = \frac{\pi b^2}{a^2} \int_{-a}^a (a^2 - x^2) dx = \frac{\pi b^2}{a^2} \left( a^2 x - \frac{x^3}{3} \right) \Big|_{-a}^a = \frac{4}{3} \pi a b^2$

$$x^2 = \frac{a^2}{b^2} (b^2 - y^2)$$

$$V_y = \pi \int_{-b}^b x^2 dy = \frac{\pi a^2}{b^2} \int_{-b}^b (b^2 - y^2) dy = \frac{\pi a^2}{b^2} \left( b^2 y - \frac{y^3}{3} \right) \Big|_{-b}^b = \frac{4}{3} \pi a^2 b$$

$$3a \quad a = b = R \quad V_L = \frac{4}{3} R^3$$


---

$$6.15. \quad V_y = \pi \int_0^1 \arcsin^2 y dy = \pi I$$

$$3a \quad I: \quad u = \arcsin^2 y, \quad dv = dy, \quad du = 2 \arcsin y \cdot \frac{dy}{\sqrt{1-y^2}}$$

$$I = y \arcsin y \Big|_0^1 - 2 \int_0^1 \arcsin y \cdot \frac{y dy}{\sqrt{1-y^2}} = \frac{\pi^2}{4} - 2J$$

$$3a \quad J: \quad u = \arcsin y, \quad dv = \frac{y dy}{\sqrt{1-y^2}}, \quad du = \frac{dy}{\sqrt{1-y^2}}, \quad v = -\sqrt{1-y^2}$$

$$J = -\sqrt{1-y^2} \arcsin y \Big|_0^1 + \int_0^1 \sqrt{1-y^2} \cdot \frac{dy}{\sqrt{1-y^2}} = 1$$

$$I = \frac{\pi^2}{4} - 2, \quad V_y = \frac{\pi}{4} (\pi^2 - 8)$$

Друго решење.

$$\arcsin y = t, \quad y = \sin t, \quad dy = \cos t dt$$

$$I = \int_0^{\pi/2} t^2 \cos t dt$$

Треће решење.

$$V_y = \left( \frac{\pi}{2} \right)^2 \pi \cdot 1 - V_y^*$$

$$V_y^* = 2\pi \int_0^{\pi/2} x \cdot \sin x dx = 2\pi$$


---

$$6.16. \quad V_y = 2^2 \pi e^2 - \pi \int_1^{e^2} \ln^2 y dy$$

$$V_y = 2\pi(e^2 + 1)$$


---

$$6.17. \quad V_x = V_1 - V_2, \quad V_1 = r^2 \pi H = 2^2 \pi \cdot 2$$

$$V_2 = \pi \int_0^2 2x dx = \pi x^2 \Big|_0^2 = 4\pi$$

$$V_x = 8\pi - 4\pi = 4\pi$$


---

$$6.18. \quad V_x = V_1 + V_2$$

$$V_1 = \pi \int_0^1 y^2 dx = \pi \int_0^1 x^2 dx = \frac{1}{3} \pi$$

$$V_2 = \pi \int_1^3 \frac{1}{x^2} dx = \frac{2}{3} \pi$$

$$V_x = \pi$$


---

$$6.19. \quad 2x - x^2 = y, \quad x^2 - 2x + y = 0, \quad x_{1/2} = 1 \pm \sqrt{1-y}$$

$$V_y = V_1 - V_2$$

$$V_1 = \pi \int_0^1 (1 + \sqrt{1-y})^2 dy = \frac{17}{6}\pi$$

$$V_2 = \pi \int_0^1 (1 - \sqrt{1-y})^2 dy = \frac{1}{6}\pi$$

$$V_y = \frac{8}{3}\pi$$

Друго решење.

$$V_y = 2\pi \int_0^2 x(2x - x^2) dx = 2\pi \int_0^2 (2x^2 - x^3) dx = 2\pi \left( \frac{2}{3}x^3 - \frac{x^4}{4} \right) \Big|_0^2 = \frac{8}{3}\pi$$

---

$$6.20. V_x = \pi \int_0^2 (2x - x^2)^2 dx$$

$$V_x = \pi \int_0^2 (4x^2 - 4x^3 + x^4) dx$$

$$V_x = \frac{16}{15}\pi$$

---

$$6.21. V = 2\pi \int_0^{r/2} (r^2 - (x-r)^2) dx$$

$$V = \frac{5}{12}\pi^3$$

---

$$6.22. V_y = \pi \int_0^4 \left( y - \frac{y^4}{64} \right) dy$$

$$V_y = \frac{24}{5}\pi$$

---

$$6.23. y' = 3x^2, \quad y'^2 = 9x^4$$

$$P_x = 2\pi \int_0^1 x^3 \sqrt{1+9x^4} dx$$

$$P_x = \frac{2\pi}{36} \int_0^1 (1+9x^4)^{1/2} d(1+9x^4)$$

$$P_x = \frac{\pi}{18} \cdot \frac{2}{3} (1+9x^4)^{3/2} \Big|_0^1$$

$$P_x = \frac{\pi}{27} (10^{3/2} - 1)$$

---

$$6.24. y' = 2x - \frac{1}{8x}, \quad 1 + y'^2 = 1 = 4x^2 + \frac{1}{64x^2} - \frac{1}{2} = \left( 2x + \frac{1}{8x} \right)^2$$

$$P_x = 2\pi \int_{\sqrt{e}}^e \left( x^2 - \frac{\ln x}{8} \right) \left( 2x + \frac{1}{8x} \right) dx$$

$$P_x = 2\pi \int_{\sqrt{e}}^e \left( 2x^3 + \frac{x}{8} - \frac{1}{4}x \ln x - \frac{1}{64} \frac{\ln x}{x} \right) dx$$

$$\int x \ln x dx = \frac{x^2}{2} \ln x - \frac{x^2}{4}$$

$$P_x = ??$$

---

$$6.25. V_x = V_1 - V_2 = \int_0^1 \pi(x - x^4) dx = \frac{3}{10} \pi$$

$$P_x = P_1 + P_2$$

$$3a P_1: \quad x = y^2, \quad y = \sqrt{x}, \quad y' = \frac{1}{2\sqrt{x}}$$

$$P_1 = 2\pi \int_0^1 \sqrt{x} \sqrt{1 + \frac{1}{4x}} dx = \pi \int_0^1 \sqrt{4x + 1} dx$$

$$P_1 = \frac{\pi}{6} (4x + 1)^{3/2} \Big|_0^1 = \frac{\pi}{6} (5\sqrt{5} - 1)$$

$$3a P_2: \quad y = x^2, \quad y' = 2x$$

$$P_2 = 2\pi \int_0^1 x^2 \sqrt{1 + 4x^2} dx = \dots = \frac{9\sqrt{5}}{16} \pi - \frac{1}{32} \pi \ln(2 + \sqrt{5})$$

$$\text{Додатак} - I = \int_0^1 x^2 \sqrt{1 + 4x^2} dx$$

$$u = x, \quad dv = x \sqrt{1 + 4x^2} dx, \quad du = dx, \quad v = \frac{1}{12} (1 + 4x^2)^{3/2}$$

$$I = \frac{x}{12} (1 + 4x^2)^{3/2} \Big|_0^1 - \frac{1}{12} \int_0^1 (1 + 4x^2)^{3/2} dx$$

$$I = \frac{5}{12} \sqrt{5} - \frac{1}{12} \int_0^1 (1 + 4x^2) \sqrt{1 + 4x^2} dx = \frac{5}{12} \sqrt{5} - \frac{1}{12} \int_0^1 \sqrt{1 + 4x^2} dx - \frac{4}{12} I$$

$$I = \frac{5}{16} \sqrt{5} - \frac{1}{16} J, \quad J = \int_0^1 \sqrt{1 + 4x^2} dx$$

$$u = \sqrt{1 + 4x^2}, \quad dv = dx, \quad du = \frac{4x dx}{\sqrt{1 + 4x^2}}, \quad v = x$$

$$J = x \sqrt{1 + 4x^2} \Big|_0^1 - 4 \int_0^1 \frac{x^2 dx}{\sqrt{1 + 4x^2}} = \sqrt{5} - J + \frac{1}{2} \ln(2x + \sqrt{1 + 4x^2})$$

$$J = \frac{\sqrt{5}}{2} + \frac{1}{4} \ln(2 + \sqrt{5}), \quad I = \frac{9}{16} \sqrt{5} - \frac{1}{64} \ln(2 + \sqrt{5})$$

$$6.26. \quad y' = \sin x, \quad 1 + f'^2 = 1 + \cos^2 x$$

$$P = 2\pi \int_0^\pi \sin x \sqrt{1 + \cos^2 x} dx = -2\pi \int_0^\pi \sqrt{1 + \cos^2 x} d(\cos x)$$

$$\cos x = t, \quad P = -2\pi \int_{-1}^1 \sqrt{1 + t^2} dt = 2\pi \int_0^1 \sqrt{1 + t^2} dt$$

$$P = 4\pi \int_0^1 \sqrt{1 + t^2} dt = 4\pi I, \quad I = \int_0^1 \sqrt{1 + t^2} dt$$

$$I = \int_0^1 \frac{1 + t^2}{\sqrt{1 + t^2}} dt = \int_0^1 \frac{dt}{\sqrt{1 + t^2}} + J, \quad 3a J: \quad u = t, \quad dv = \frac{t dt}{\sqrt{1 + t^2}}$$

$$J = \int_0^1 \frac{t^2 dt}{\sqrt{1 + t^2}} = t \sqrt{1 + t^2} \Big|_0^1 - I, \quad 2I = \ln(1 + \sqrt{2}) + \sqrt{2}$$

$$P = 4\pi I = 2\pi \ln(1 + \sqrt{2}) + 2\sqrt{2}\pi = \pi(\ln(3 + 2\sqrt{2}) + 2\sqrt{2})$$

$$8.1. \quad I = \int_0^1 x^2 dx \cdot \int_0^1 \frac{dy}{1 + y^2}$$

$$I = \frac{x^3}{3} \Big|_0^1 \cdot \arctan y \Big|_0^1 dy$$



$$I = 1 \cdot \frac{\pi}{4} = \frac{\pi}{4}$$


---

8.2.  $\mathcal{D} = \{(x, y) : 0 \leq x \leq 4, x/2 \leq y \leq x\}$

$$\begin{aligned} \iint_{\mathcal{D}} (x^3 + y^3) dx dy &= \int_0^4 dx \int_{x/2}^x (x^3 + y^3) dy \\ &= \int_0^4 \left[ x^3 y + \frac{y^4}{4} \right] \Big|_{y=x/2}^{y=x} dx \\ &= \int_0^4 \left[ x^3(x - x/2) + \frac{1}{4}(x^4 - x^4/64) \right] dx \\ &= \frac{47}{64} \int_0^4 x^4 dx \\ &= \frac{752}{5} \end{aligned}$$


---

8.3.  $I = \int_0^1 \left[ \int_0^1 \frac{dy}{(x+y+1)^2} \right] dx$

$$I = \int_0^1 \left[ \frac{-1}{x+y+1} \Big|_0^1 \right] dx$$

$$I = \int_0^1 \left[ \frac{-1}{x+2} + \frac{1}{x+1} \right] dx$$

$$I = -\ln(x+2) + \ln(x+1) \Big|_0^1 = \ln \frac{x+1}{x+2} \Big|_0^1$$

$$I = \ln \frac{4}{3}$$


---

8.4.  $I = \int_0^2 \left[ \int_{x^2/2}^{\sqrt{2x}} (x+y) dy \right] dx$

$$I = \int_0^2 x \cdot \left( y \Big|_{x^2/2}^{\sqrt{2x}} \right) dx + \int_0^2 \left( \frac{y^2}{2} \Big|_{x^2/2}^{\sqrt{2x}} \right) dx$$

$$I = \int_0^2 \left( x\sqrt{2x} - \frac{1}{2}x^3 \right) dx + \int_0^2 \left( x - \frac{1}{4}x^4 \right) dx$$

$$I = \frac{8}{5}$$


---

8.5. Најпре по  $x$ , па по  $y$ !

$$I = \int_1^3 \left[ \int_{y-1}^y (x^2 + y^2) dx \right] dy = \int_1^3 I(y) dy$$

$$I(y) = \int_{y-1}^y (x^2 + y^2) dx = \frac{x^3}{3} \Big|_{y-1}^y + y^2 \cdot x \Big|_{y-1}^y$$

$$I(y) = \frac{y^3}{3} - \frac{1}{3}(y-1)^3 + y^2(y-y+1) = 2y^2 - y + \frac{1}{3}$$

$$I = \int_1^3 \left( 2y^2 - y + \frac{1}{3} \right) dy$$

$$I = 14$$

---


$$8.6. I = \int_0^{\sqrt{3}/2} y \cdot I(y) dy, \quad I(y) = \int_{1-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x dx$$

$$I(y) = \frac{x^2}{2} \Big|_{1-\sqrt{1-y^2}}^{\sqrt{1-y^2}}$$

$$I(y) = \frac{1}{2}(1-y^2) - \frac{1}{2}(1+1-y^2-2\sqrt{1-y^2}) = -\frac{1}{2} + \sqrt{1-y^2}$$

$$I = -\frac{1}{2} \int_0^{\sqrt{3}/2} (y - 2y\sqrt{1-y^2}) dy$$

$$I = \frac{5}{48}$$


---

$$8.7. I = \int_0^1 x^2 \cdot I(x) dx, \quad I(x) = \int_0^{\sqrt[3]{1-x^3}} y^2 \sqrt{1-x^3-y^3} dy$$

$$I(x) = \frac{1}{3} \cdot \frac{2}{3} (1-x^3-y^3)^{3/2} \Big|_0^{\sqrt[3]{1-x^3}}$$

$$I(x) = \frac{2}{9} (1-x^3)^{3/2}$$

$$I = \frac{2}{9} \int_0^1 x^2 (1-x^3)^{3/2} dx = \frac{2}{9} \cdot \frac{1}{3} \int_0^1 (1-x^3)^{3/2} d(1-x^3)$$

$$I = \frac{4}{135}$$


---

$$8.8. |y-x^2| = \begin{cases} y-x^2, & y \geq x^2 \\ x^2-y, & y < x^2 \end{cases} = \begin{cases} y-x^2, & (x,y) \in \mathcal{D}_1 \\ x^2-y, & (x,y) \in \mathcal{D}_2 \end{cases}$$

$$I = \iint_{\mathcal{D}_1} f(x,y) dx dy + \iint_{\mathcal{D}_2} f(x,y) dx dy = I_1 + I_2$$

$$I_1 = \int_0^{\sqrt{\pi/2}} x dx \int_{x^2}^{\pi} \sin(y-x^2) dy = \int_0^{\sqrt{\pi/2}} x(\cos x^2 + 1) dx$$

$$I_1 = \frac{1}{2} \sin x^2 + \frac{x^2}{2} \Big|_0^{\sqrt{\pi/2}} = \frac{1}{2} + \frac{\pi}{4}$$

$$I_2 = \int_0^{\sqrt{\pi/2}} x dx \int_0^{x^2} \sin(x^2-y) dy = \int_0^{\sqrt{\pi/2}} x(1-\cos x^2) dx$$

$$I_2 = \frac{x^2}{2} - \frac{1}{2} \sin x^2 \Big|_0^{\sqrt{\pi/2}} = \frac{\pi}{4} - \frac{1}{2}$$

$$I = I_1 + I_2 = \frac{1}{2} + \frac{\pi}{4} + \frac{\pi}{4} - \frac{1}{2} = \frac{\pi}{2}$$


---

$$8.9. x+y=u, x-y=v, \quad x = \frac{u+v}{2}, y = \frac{u-v}{2}$$

$$x+y=1 \rightarrow u=1, \quad x+y=3 \rightarrow u=3,$$

$$2x-y=2 \rightarrow u+3v=4, \quad 2x-y=-1 \rightarrow u+3v=-2$$

$$\mathcal{G}: u=1, u=3, u+3v=-2, u+3v=4$$

$$J = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix}$$

$$I = \int_1^3 du \int_{-(u+2)/3}^{(4-u)/3} uv|J|dv = \frac{1}{2} \int_1^3 u \left[ \frac{v^2}{2} \Big|_{-(u+2)/3}^{(4-u)/3} \right] du$$

$$I = -\frac{14}{9}$$

Друго решење.

$$x + y = u, \quad 2x - y = v, \quad x = \frac{u+v}{3}, \quad y = \frac{2u-v}{3}, \quad J = -\frac{1}{3}$$

$$u \in [1, 3], \quad v \in [-1, 2]$$

$$I = \int_1^3 du \int_{-1}^2 \frac{u}{3}(2v-u) \cdot \frac{1}{3} dv$$

$$I = -\frac{14}{9}$$


---

$$8.10. \quad u = x - y, \quad v = x + y, \quad x = \frac{u+v}{2}, \quad y = \frac{v-u}{2}, \quad J = \frac{1}{2}$$

$$\mathcal{G}: v = 1, \quad u = v, \quad u = -v$$

$$I = \frac{1}{2} \int_0^1 dv \int_{-v}^v \cos \frac{u}{v} du$$

$$I = \frac{1}{2} \int_0^1 \left( v \sin \frac{u}{v} \right) \Big|_{-v}^v dv$$

$$I = \sin 1 \cdot \int_0^1 v dv = \frac{1}{2} \sin 1$$


---

$$8.11. \quad \frac{y}{x} = u, \quad xy = v, \quad x = \sqrt{\frac{v}{u}}, \quad y = \sqrt{uv}, \quad (u, v) \in [\alpha, \beta] \times [0, a]$$

$$\frac{D(u, v)}{D(x, y)} = \begin{vmatrix} -y/x^2 & 1/x \\ y & x \end{vmatrix} = -\frac{y}{x}, \quad \frac{D(x, y)}{D(u, v)} = -\frac{1}{2} \cdot \frac{x}{y} = -\frac{1}{2u}$$

$$I = \frac{1}{2} \int_{\alpha}^{\beta} \frac{du}{u} \int_0^a \sin \frac{\pi v}{a} dv$$

$$I = \frac{1}{2} \ln \frac{\beta}{\alpha} \left[ -\frac{a}{\pi} \cos \frac{\pi v}{a} \Big|_0^a \right] = \frac{a}{\pi} \ln \frac{\beta}{\alpha}$$


---

$$8.12. \quad x = \rho \cos \varphi, \quad y = \rho \sin \varphi, \quad \rho \in [0, 1], \quad \varphi \in [0, \pi/2]$$

$$\iint_D \sqrt{a^2 - x^2 - y^2} dx dy = \int_0^{\pi/2} d\varphi \int_0^1 \sqrt{a^2 - \rho^2} \rho d\rho = \frac{1}{3} a^3 \int_0^{\pi/2} d\varphi = \frac{a^3}{6} \pi$$


---

$$8.13. \quad x = \rho \cos \varphi, \quad y = \rho \sin \varphi, \quad J = \rho$$

$$\mathcal{G} = [0, 2\pi] \times [\pi/3, \pi]$$

$$I = \int_0^{2\pi} d\varphi \int_{\pi/3}^{\pi} \sin \rho d\rho$$

$$I = 2\pi \cdot (-\cos \varphi) \Big|_{\pi/3}^{\pi}$$

$$I = 3\pi$$


---

$$8.14. I = \int_0^{2\pi} d\varphi \int_0^R (1 - 2\rho \cos \varphi - 3\rho \sin \varphi) \rho d\rho$$

$$I = \int_0^{2\pi} d\varphi \left( \frac{\rho^2}{2} - \frac{2}{3}\rho^3 \cos \varphi - \rho^3 \sin \varphi \right) \Big|_0^R$$

$$I = \frac{R^2}{6} (3\varphi - 4R \sin \varphi + 6R \cos \varphi) \Big|_0^{2\pi}$$

$$I = \pi R^2$$


---

$$8.15. x = \rho \cos \varphi, y = \rho \sin \varphi, \quad \rho \in [0, \pi/4], \varphi \in [4 \cos \varphi, 8 \cos \varphi]$$

$$I = \iint_G \frac{\rho d\rho d\varphi}{\rho^4} = \int_0^{\pi/4} d\varphi \int_{4 \cos \varphi}^{8 \cos \varphi} \frac{d\rho}{\rho^3}$$

$$I = -\frac{1}{2} \int_0^{\pi/4} d\varphi \cdot \frac{1}{\rho^2} \Big|_{4 \cos \varphi}^{8 \cos \varphi} = -\frac{1}{2} \int_0^{\pi/4} \left( \frac{1}{64 \cos^2 \varphi} - \frac{1}{16 \cos^2 \varphi} \right) d\varphi$$

$$I = -\frac{1}{2} \left( \frac{1}{64} - \frac{1}{16} \right) \int_0^{\pi/4} \frac{d\varphi}{\cos^2 \varphi}$$

$$I = -\frac{1}{2} \cdot \frac{1-4}{64} \cdot \tan \varphi \Big|_0^{\pi/4} = \frac{3}{128}$$


---

$$8.16. \rho \in [0, R \cos \varphi], \varphi \in [0, \pi/2]$$

$$I = \int_0^{\pi/2} d\varphi \int_0^{R \cos \varphi} \sqrt{R^2 - \rho^2} \rho d\rho = -\frac{1}{3} \int_0^{\pi/2} d\varphi \sqrt{(R^2 - \rho^2)^3} \Big|_0^{R \cos \varphi}$$

$$I = -\frac{1}{3} \int_0^{\pi/2} \left( \sqrt{(R^2 - R^2 \cos^2 \varphi)^3} - R^3 \right) d\varphi$$

$$I = -\frac{R^3}{3} \int_0^{\pi/2} (\sin^3 \varphi - 1) d\varphi = -\frac{R^3}{3} J$$

$$J = \int_0^{\pi/2} ((1 - \cos^2 \varphi) \sin \varphi - 1) d\varphi = -\cos \varphi + \frac{1}{3} \cos^3 \varphi - \varphi \Big|_0^{\pi/2} = -\frac{\pi}{2} + \frac{2}{3}$$

$$I = \frac{R^3}{6} \left( \pi - \frac{4}{3} \right)$$


---

$$8.17. \rho \in [1, 3], \varphi \in [\pi/6, \pi/3], \quad \frac{y}{x} = \tan \varphi, \arctan \frac{y}{x} = \varphi$$

$$I = \int_{\pi/6}^{\pi/3} \varphi d\varphi \int_1^3 \rho d\rho$$

$$I = \frac{\varphi^2}{2} \Big|_{\pi/6}^{\pi/3} \cdot \frac{\rho^2}{2} \Big|_1^3$$

$$I = \frac{\pi^2}{6}$$


---

$$8.18. x = a\rho \cos \theta, y = b\rho \sin \theta$$

$$\rho \in [1, 2], \theta \in [0, \pi/2], \quad J = ab\rho$$

$$I = ab \int_0^{\pi/2} d\theta \int_1^2 \sqrt{4 - \rho^2} \rho d\rho$$

$$I = -\frac{1}{2} \cdot ab \cdot \frac{\pi}{2} \int_1^2 \sqrt{4 - \rho^2} d(4 - \rho^2)$$

$$I = -ab \cdot \frac{\pi}{2} \cdot \frac{1}{3} \sqrt{(4 - \rho^2)^3} \Big|_1$$

$$I = \frac{\sqrt{3}}{2} ab\pi$$


---

9.1.  $V = \iint_{\mathcal{D}} z \, dx dy$ ,  $\mathcal{D} = \{(x, y) : x \geq 0, y \geq 0, y \leq 3 - x\}$

$$V = \int_0^3 dy \int_0^{3-y} (4x^2 + 2y^2 + 1) dx = \int_0^3 I(y) dy$$

$$\begin{aligned} I(y) &= \int_0^{3-y} (4x^2 + 2y^2 + 1) dx = \left[ \frac{4}{3}x^3 + 2xy^2 + x \right] \Big|_{x=0}^{x=3-y} \\ &= \frac{4}{3}(3-y)^3 + 2(3-y)y^2 + (3-y) \\ &= 39 - 37y + 18y^2 - \frac{10}{3}y^3 \end{aligned}$$

$$\begin{aligned} V &= \int_0^3 \left( 39 - 37y + 18y^2 - \frac{10}{3}y^3 \right) dy \\ &= \left[ 39y - \frac{37}{2}y^2 + 6y^3 - \frac{10}{12}y^4 \right] \Big|_0^3 \\ &= 45 \end{aligned}$$


---

9.2.  $V = \iint_{\mathcal{D}} xy \, dx dy$ ,  $x = 1 + \rho \cos \varphi$ ,  $y = 1 + \rho \sin \varphi$ ,  $\rho \in [0, 1]$ ,  $J = \rho$

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^1 \rho(1 + \rho \cos \varphi)(1 + \rho \sin \varphi) d\rho d\varphi \\ &= \int_0^{2\pi} \int_0^1 \rho d\rho d\varphi + \int_0^{2\pi} \int_0^1 \rho^2 (\sin \varphi + \cos \varphi + \rho \sin \varphi \cos \varphi) d\rho d\varphi \\ &= \int_0^{2\pi} d\varphi \cdot \int_0^1 \rho d\rho \\ &= \pi \end{aligned}$$

јер је

$$\int_0^{2\pi} \sin \varphi d\varphi = \int_0^{2\pi} \cos \varphi d\varphi = \int_0^{2\pi} \sin \varphi \cos \varphi d\varphi = 0.$$


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9.3.  $V = \iint_{\mathcal{D}} z \, dx dy$ ,  $\mathcal{D} = \{(x, y) : x \geq 0, y \geq 0, y \leq a^2 - x^2\}$

$$V = \int_0^a dx \int_0^{\sqrt{a^2-x^2}} (a^2 - x^2) dy = \int_0^a (a^2 - x^2) I(x) dx$$

$$I(x) = \int_0^{\sqrt{a^2-x^2}} dy = y \Big|_0^{\sqrt{a^2-x^2}} = \sqrt{a^2 - x^2}$$

$$V = \int_0^a (a^2 - x^2) \sqrt{a^2 - x^2} dx = \int_0^a (a^2 - x^2)^{3/2} dx = a^3 \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{\pi}{2} = \frac{3}{16} a^4 \pi$$

(видети Збирка, зад.83 (2), стр.87)

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$$9.4. \quad V = \iint_{\mathcal{D}} z dx dy, \quad \mathcal{D} = \{(x, y) : x^2 + y^2 \leq 2x, x^2 + y^2 \leq 2y\} = \mathcal{D}_1 \cup \mathcal{D}_2$$

$$x = \rho \cos \varphi, \quad y = \rho \sin \varphi, \quad K_1 : x^2 + y^2 = 2x, \quad \rho = 2 \cos \varphi, \quad K_2 : x^2 + y^2 = 2y, \quad \rho = 2 \sin \varphi$$

$$\begin{aligned} V &= \iint_{\mathcal{D}} (x + 2y) dx dy \\ &= \int_0^{\pi/4} (\cos \varphi + 2 \sin \varphi) d\varphi \int_0^{2 \sin \varphi} \rho^2 d\rho + \int_{\pi/4}^{\pi/2} (\cos \varphi + 2 \sin \varphi) d\varphi \int_0^{2 \cos \varphi} \rho^2 d\rho \\ &= \dots \\ &= \frac{3}{4}\pi - \frac{3}{2} \end{aligned}$$

$$9.5. \quad V = V_1 - V_2$$

$$V_1 = \iint_{\mathcal{D}} 2(x^2 + y^2) dx dy, \quad V_2 = \iint_{\mathcal{D}} (x^2 + y^2) dx dy$$

$$\mathcal{D} = \{(x, y) \mid 0 \leq x \leq 1, x \leq y \leq \sqrt{x}\}$$

$$V = \iint_{\mathcal{D}} (x^2 + y^2) dx dy = \int_0^1 dx \int_x^{\sqrt{x}} (x^2 + y^2) dy$$

$$V = \frac{3}{35}$$

$$9.6. \quad x = 1 + \rho \cos \varphi, \quad y = \rho \sin \varphi$$

$$V = \int_0^{\pi} d\varphi \int_0^1 (2 + 2\rho \cos \varphi + \rho \sin \varphi) \rho d\rho$$

$$V = \int_0^{\pi} \left(1 + \frac{2}{3} \cos \varphi + \frac{1}{3} \sin \varphi\right) d\varphi = \pi + \frac{2}{3}$$

Друго решење.

$$x = \rho \cos \varphi, \quad y = \rho \sin \varphi$$

$$V = \int_0^{\pi/2} d\varphi \int_0^{2 \cos \varphi} (2 \cos \varphi + \sin \varphi) \rho^2 d\rho$$

$$V = \frac{16}{3} \int_0^{\pi/2} \cos^4 \varphi d\varphi - \frac{8}{3} \int_0^{\pi/2} \cos^3 \varphi d(\cos \varphi) = \frac{16}{3} \cdot \frac{3\pi}{16} - \frac{8}{3} \cdot \frac{-1}{4} = \pi + \frac{2}{3}$$

$$9.7. \quad x = 1 + \rho \cos \varphi, \quad y = \rho \sin \varphi, \quad \rho \in [0, 1], \quad \varphi \in [0, 2\pi]$$

$$V = \iint_{\mathcal{D}} \frac{x^2 + y^2}{2} dx dy = \int_0^{2\pi} d\varphi \int_0^1 \frac{(1 + \rho \cos \varphi)^2 + (\rho \sin \varphi)^2}{2} \cdot \rho d\rho$$

$$V = \int_0^{2\pi} d\varphi \int_0^1 \left(\frac{\rho}{2} + \rho^2 \cos \varphi + \frac{\rho^3}{2}\right) d\rho$$

$$V = \int_0^{2\pi} d\varphi \left(\frac{\rho^2}{4} + \frac{\cos \varphi}{3} \rho^3 + \frac{\rho^4}{8}\right) \Big|_0^1$$

$$V = \int_0^{2\pi} \left(\frac{3}{8} + \frac{\cos \varphi}{3}\right) d\varphi = \left(\frac{3}{8}\varphi + \frac{\sin \varphi}{3}\right) \Big|_0^{2\pi} = \frac{3}{4}\pi$$

Друго решење.

$$x = \rho \cos \varphi, \quad y = \rho \sin \varphi, \quad \rho \in [0, 2 \cos \varphi], \quad \varphi \in [-\pi/2, \pi/2]$$

$$V = \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2 \cos \varphi} \frac{\rho^3}{2} d\rho = 2 \int_{-\pi/2}^{\pi/2} \cos^4 \varphi d\varphi$$

$$V = 2 \int_{-\pi/2}^{\pi/2} \left( \frac{1 + \cos 2\varphi}{2} \right)^2 d\varphi$$

$$V = 2 \left( \frac{3}{8}\varphi + \frac{\sin 2\varphi}{4} + \frac{\sin 4\varphi}{32} \right) \Big|_{-\pi/2}^{\pi/2} = \frac{3}{4}\pi$$

9.8.  $P = \iint_{\mathcal{D}} dx dy, \mathcal{D} = \left\{ (x, y) : \frac{1}{a}(x-a)^2 \leq y \leq \sqrt{a^2 - x^2} \right\}$

$$P = \int_0^a dx \int_{(x-a)^2/a}^{\sqrt{a^2-x^2}} dy$$

$$= \frac{a^2}{4}\pi - \frac{1}{a} \left[ \int_0^a x^2 dx - 2a \int_0^a x dx + a^2 \int_0^a dx \right]$$

$$= \frac{a^2}{4}\pi - \frac{1}{a} \cdot \frac{x^3}{3} \Big|_0^a + x^2 \Big|_0^a - ax \Big|_0^a$$

$$= \frac{a^2}{12}(3\pi - 4)$$

За  $\int_0^a \sqrt{a^2 - x^2} = \frac{a^2}{4}\pi$  видети: Збирка, зад.58, стр.86.

9.9.  $\mathcal{D} = \{(x, y) : a/2 \leq y \leq a, a - y \leq x \leq \sqrt{a^2 - y^2}\}$

$$\int_{a/2}^a I(y) dy, \quad I(y) = \int_{a-y}^{\sqrt{a^2-y^2}} dx = \sqrt{a^2 - y^2} - a + y$$

Како је

$$P = \int \sqrt{a^2 - x^2} dx = \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2} + C$$

(видети Збирка, зад.76, стр.76), то је

$$P = \int_{a/2}^a \sqrt{a^2 - y^2} dy - a \int_{a/2}^a + \int_{a/2}^a y dy$$

$$= \frac{a^2}{2} \arcsin \frac{y}{a} \Big|_{a/2}^a - a(a - a/2) + \frac{y^2}{2} \Big|_{a/2}^a$$

$$= \frac{a}{2} \left( \frac{\pi}{2} - \frac{\pi}{6} \right) + 0 - \frac{a^2}{8} \sqrt{3} - \frac{a^2}{2} + \frac{a^2}{2} - \frac{a^2}{8}$$

$$= \frac{a^2}{6}\pi - \frac{a^2}{8}(\sqrt{3} + 1)$$

9.10.  $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2,$

$$\mathcal{D}_1 = \{(x, y) : 0 \leq x \leq a/2, x^2/a \leq y \leq 2a\}, \mathcal{D}_2 = \{(x, y) : a/2 \leq x \leq a, x^2/a \leq y \leq a^2/x\}$$

$$P = \int_0^{a/2} dx \int_{x^2/a}^{2a} dy + \int_{a/2}^a dx \int_{x^2/a}^{a^2/x} dy = \dots = \frac{23}{24}a^2 + a^2 \ln 2$$

9.11. Из  $\frac{y^2 + b^2}{2b} = \frac{y^2 + a^2}{2a}$  следи  $x = \frac{a+b}{2}, y = \pm\sqrt{ab}$

$$P = 2 \iint_{\mathcal{D}} dx dy, \mathcal{D} = \left\{ 0 \leq y \leq \sqrt{ab}, x_1 \leq x \leq x_2 \right\}, x_1 = \frac{y^2 + b^2}{2b}, x_2 = \frac{y^2 + a^2}{2a}$$

$$P = 2 \int_0^{\sqrt{ab}} dy \int_{x_1}^{x_2} dx = 2 \int_0^{\sqrt{ab}} \left( \frac{y^2 + b^2}{2b} - \frac{y^2 + a^2}{2a} \right) dy = \dots = \frac{2}{3}(a-b)\sqrt{ab}$$

---

9.12.  $x = \rho \cos \varphi$ ,  $y = \rho \sin \varphi$

$$K_1 : (x - a)^2 + y^2 = a^2, \quad x^2 - 2ax + y^2 = 0, \quad x^2 + y^2 = 2ax, \quad r^2 = 2ar \cos \varphi, \quad r = 2a \cos \varphi$$

$$K_2 : x^2 + (y - a)^2 = a^2, \quad x^2 - 2ay + y^2 = 0, \quad x^2 + y^2 = 2ay, \quad r^2 = 2ar \sin \varphi, \quad r = 2a \sin \varphi$$

$$\begin{aligned} P &= \int_0^{\pi/4} d\varphi \int_0^{2a \sin \varphi} \rho d\rho + \int_{\pi/4}^{\pi/2} d\varphi \int_0^{2a \cos \varphi} \rho d\rho \\ &= 2a^2 \int_0^{\pi/4} \sin^2 \varphi d\varphi + 2a^2 \int_{\pi/4}^{\pi/2} \cos^2 \varphi d\varphi \\ &= a^2 \left( \frac{\pi}{4} - \frac{1}{2} \right) + a^2 \left( \frac{\pi}{4} - \frac{1}{2} \right) \\ &= a^2 \left( \frac{\pi}{2} - 1 \right) \end{aligned}$$

---

9.13.  $\mathcal{D} = \{(x, y) : x^2 + y^2 \geq ax, x^2 + y^2 \leq 2ax\}$ ,  $x = \rho \cos \varphi$ ,  $y = \rho \sin \varphi$ ,  $\rho \in [a \cos \varphi, 2a \cos \varphi]$

$$\begin{aligned} P &= 2 \int_0^{\pi/2} d\varphi \int_{a \cos \varphi}^{2a \cos \varphi} \rho d\rho \\ &= \int_0^{\pi/2} \rho^2 \Big|_{a \cos \varphi}^{2a \cos \varphi} d\varphi \\ &= 3a^2 \int_0^{\pi/2} \cos^2 \varphi d\varphi \\ &= 3a^2 \int_0^{\pi/2} \frac{1 + \cos 2\varphi}{2} d\varphi \\ &= \frac{3}{4} a^2 \pi \end{aligned}$$

---

9.14.  $x = \rho \cos \varphi$ ,  $y = \rho \sin \varphi$ ,  $J = \rho$

$$\mathcal{G} : -\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{4}, \quad 0 \leq \rho \leq 2 \cos \varphi$$

$$P(\mathcal{D}) = \iint_{\mathcal{D}} dx dy = \iint_{\mathcal{G}} \rho d\varphi d\rho$$

$$P(\mathcal{D}) = 2 \int_0^{\pi/4} d\varphi \int_0^{2 \cos \varphi} \rho d\rho = 2 \cdot \frac{\pi}{4} \cdot \frac{\rho^2}{2} \Big|_0^{2 \cos \varphi}$$

$$P(\mathcal{D}) = 4 \int_0^{\pi/4} \cos^2 \varphi d\varphi = \frac{\pi}{2} + 1$$

---

9.15.  $P(\mathcal{D}) = \iint_{\mathcal{D}} dx dy$

$$u = \frac{x}{a} + \frac{y}{b}, \quad v = \frac{x}{a} - \frac{y}{b}$$

$$\mathcal{G} : v = u^2, \quad v = u, \quad J = -\frac{ab}{2}$$

$$P(\mathcal{D}) = \frac{ab}{2} \int_0^1 du \int_{u^2}^u dv = \frac{ab}{2} \int_0^1 (u - u^2) du$$



$$P(\mathcal{D}) = \frac{ab}{2} \cdot \frac{1}{6} = \frac{ab}{12}$$


---

9.16.  $z'_x = 2x, \quad z'_y = 2y, \quad 1 + z'^2_x + z'^2_y = 1 + 4x^2 + 4y^2$

$$P = \iint_{\mathcal{D}} \sqrt{1 + 4x^2 + 4y^2} \, dx dy$$

$x = \rho \cos \varphi, \quad y = \rho \sin \varphi, \quad J = \rho, \quad 1 + 4x^2 + 4y^2 = 1 + 4\rho^2$

$$P = \int_0^{2\pi} d\varphi \int_0^1 \sqrt{1 + 4\rho^2} \rho d\rho$$

$$P = \frac{\pi}{6} (5^{3/2} - 1)$$


---

9.17.  $z = \sqrt{a^2 - x^2 - y^2}, \quad \mathcal{D} : (x - a/2)^2 + y^2 = a^2/4$

$$z'_x = \frac{-x}{\sqrt{a^2 - x^2 - y^2}}, \quad z'_y = \frac{-y}{\sqrt{a^2 - x^2 - y^2}}$$

$$P = 2 \iint_{\mathcal{D}} \sqrt{1 + z'^2_x + z'^2_y} \, dx dy = 2 \iint_{\mathcal{D}} \frac{a \, dx dy}{\sqrt{a^2 - x^2 - y^2}}$$

$x = \rho \cos \varphi, \quad y = \rho \sin \varphi$

$\mathcal{G} : \rho = a \cos \varphi, \quad \varphi \in [0, \pi/2]$

$$P = 2 \int_0^{\pi/2} \int_0^{a \cos \varphi} \frac{a \rho d\rho}{\sqrt{a^2 - \rho^2}} d\varphi$$

$$P = 2a^2 \int_0^{\pi/2} (1 - \sin \varphi) d\varphi = (\pi - 2)a^2$$


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9.18.  $z = \sqrt{x^2 + y^2}, \quad \mathcal{D} : x^2 + y^2 \leq 2ax, \quad \mathcal{D} : (x - a)^2 + y^2 \leq a^2$

$$z'^2_x = \frac{x}{\sqrt{x^2 + y^2}}, \quad z'^2_y = \frac{y}{\sqrt{x^2 + y^2}}, \quad \sqrt{1 + z'^2_x + z'^2_y} = \sqrt{2}$$

$$\begin{aligned} P &= 2 \iint_{\mathcal{D}} \sqrt{2} \, dx dy \\ &= 2\sqrt{2} \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2a \cos \varphi} \rho d\rho \\ &= 4\sqrt{2} \int_0^{\pi/2} \frac{1}{2} \rho^2 \Big|_0^{2a \cos \varphi} d\varphi \\ &= 2\sqrt{2} \int_0^{\pi/2} 4a^2 \cos^2 \varphi d\varphi \\ &= 4a^2 \sqrt{2} \int_0^{\pi/2} (1 + \cos 2\varphi) d\varphi \\ &= 4\sqrt{2} a^2 \left( \varphi + \frac{1}{2} \sin 2\varphi \right) \Big|_0^{\pi/2} \\ &= 2\sqrt{2} a^2 \pi. \end{aligned}$$