MATEMATIKA 3

Vera & Rade

1. Diferencijalne jednačine prvog reda - osnovni pojmovi

Oznake:

- x - nezavisno promenljiva
- y - nepoznata funkcija, $y = y(x)$
- $y' = \frac{dy}{dx}$ - izvod funkcije

Opšti oblik diferencijalne jednačine prvog reda:

$$F(x, y, y') = 0 \tag{1}$$

Normalni oblik diferencijalne jednačine prvog reda:

$$y' = f(x, y) \tag{2}$$

Napomena: (1) i (2) su funkcionalne jednačine

Primer: $100g$ šećera topi se u vodi brzinom proporcionalnom nerastvorenom delu. Naći jednačinu topljenja šečera.

$q(t)$ - broj istopljenih grama u trenutku t

$q'(t)$ - brzina topljenja

$$q' = k(100 - q),\ q(0) = 0$$

Definicija: Funkcija $y = y(x)$ je **rešenje** jednačine (2) na (a, b) ako je

$$y'(x) \equiv f(x, y(x)), \quad x \in (a, b)$$

Primer:

$$q(t) = 100 - 100e^{-kt}$$

$$\begin{aligned} q' &= 100ke^{-kt} = k(100 - 100 + 100e^{-kt}) = \\ &= k(100 - q),\ t \in [0, \infty) \end{aligned}$$

Primer iz ekonomije (funkcija mortaliteta u osiguranju): Populacija se sastoji od ljudi iste starosti.

$l(x)$ - veličina populacije u u trenutku x
$l(x) \searrow$ (uzrok je mortalitet)

Brzina promene populacije proporcionalna je veličini populacije:

$$l'(x) = -\underbrace{(p + qr^x)}_{\mu(x)} l(x) \qquad (*)$$

$\mu(x) > 0$ - intenzitet smrtnosti
p, q, r - parametri

Rešenje: $\quad l(x) = Cs^x q^{r^x}$

Gomperdtz-Makeham-ov zakon smrtnosti

$-p = \ln s, \;\; -\frac{q}{\ln r} = \ln q, \;\; C -$ slobodan parametar

Domaći: Proveriti da je $l(x) = Cs^x q^{r^x}$ opšte rešenje jednačine $(*)$.

Napomena: U principu, jednačina (2) ima beskonačno rešenja

Definicija: **Opšte rešenje** j-ne (2) je eksplicitno zadata familija funkcija $y = y(x, C)$ ili implicitno zadata familija $\varphi(x, y, C) = 0$ koja identički zadovoljava (2) po x i C

Primer: $q(t) = 100 + Ce^{-kt}$
$q' = -Cke^{-kt} = k(100 - 100 - Ce^{-kt}) =$
$= k(100 - q), \;\; C \in (-\infty, \infty), \; t \in (-\infty, \infty)$

Definicija: **Partikularno rešenje** j-ne (2) je svako rešenje koje se dobija iz opšteg za fiksiranu vrednost konstante C

Primer: Za $C = -100$ dobija se partikularno rešenje $q = 100 - 100e^{-kt}$

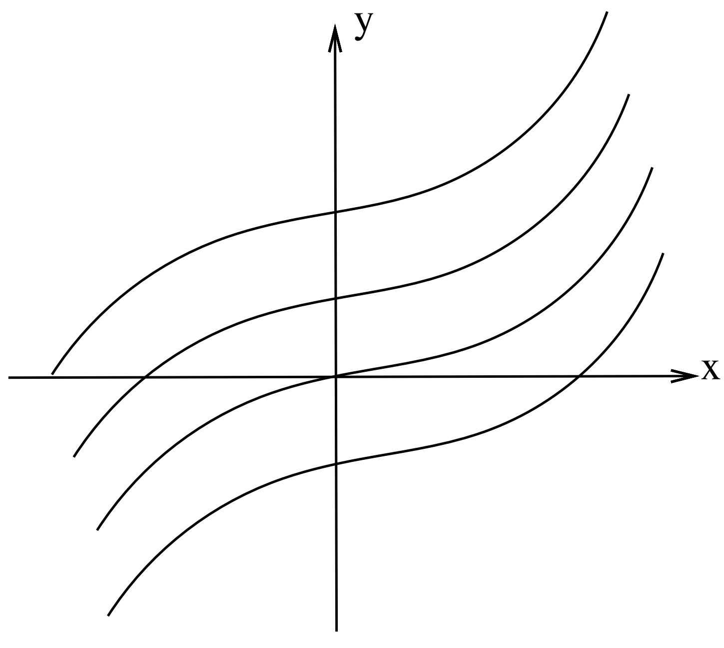
Definicija: **Singularno rešenje** j-ne (2) je rešenje koje se ne može dobiti iz opšteg ni za jednu vrednost konstante C

Pitanje: Kako doći do opšteg rešenja?

Primer: $y' = x^2 \quad (f(x,y) = x^2$ ne zavisi od $y)$

Skup svih rešenja je skup svih primitivnih funkcija f-je x^2:

$$y = \int x^2 dx = \frac{x^3}{3} + C$$



Sl. 1

Opšte rešenje $y = \frac{x^3}{3} + C$ sadrži sva rešenja.

Za $C = 0$ dobija se partikularno rešenje $y = \frac{x^3}{3}$.

Napomena: Ako je data familija $y = \frac{x^3}{3} + C$, parametar C se može eliminisati diferenciranjem:

$$y' = 3 \cdot \frac{x^2}{3} = x^2$$

Dobija se diferencijalna j-na čije je opšte rešenje polazna familija.

Ova napomenena važi u opštem slučaju:

$$y = y(x, C), \quad y' = \frac{d}{dx} y(x, C)$$

Eliminacija C iz dve veze daje diferencijalnu j-nu čije je rešenje polazna familija

$$y' = f(x, y) \longleftrightarrow y = y(x, C)$$

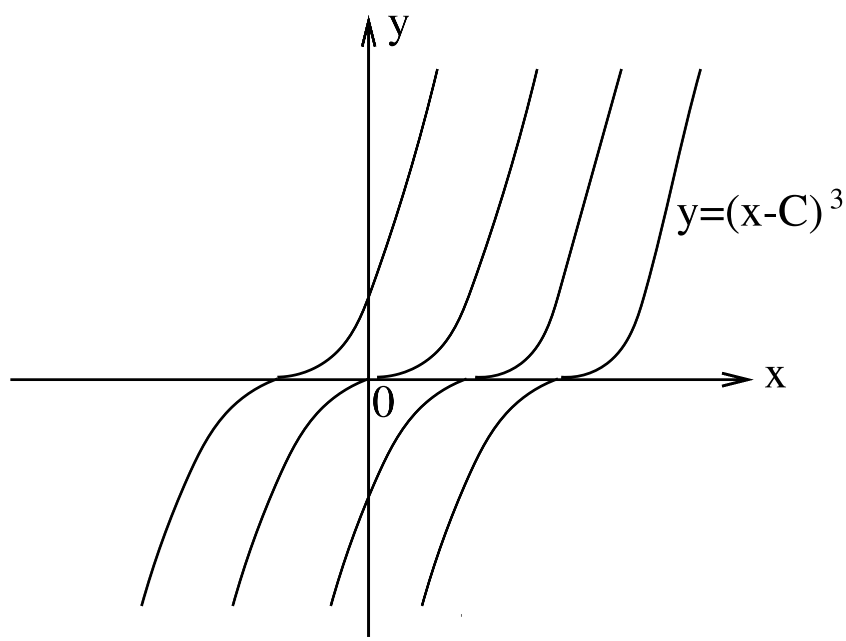
Primer:

$$y = (x - C)^3$$
$$y' = 3(x - C)^2$$

$$x - C = \sqrt[3]{y} \Rightarrow y' = 3(\sqrt[3]{y})^2$$

Dobijena je jednačina: $y' = 3\sqrt[3]{y^2}$

Opšte rešenje j-ne: $y = (x - C)^3$

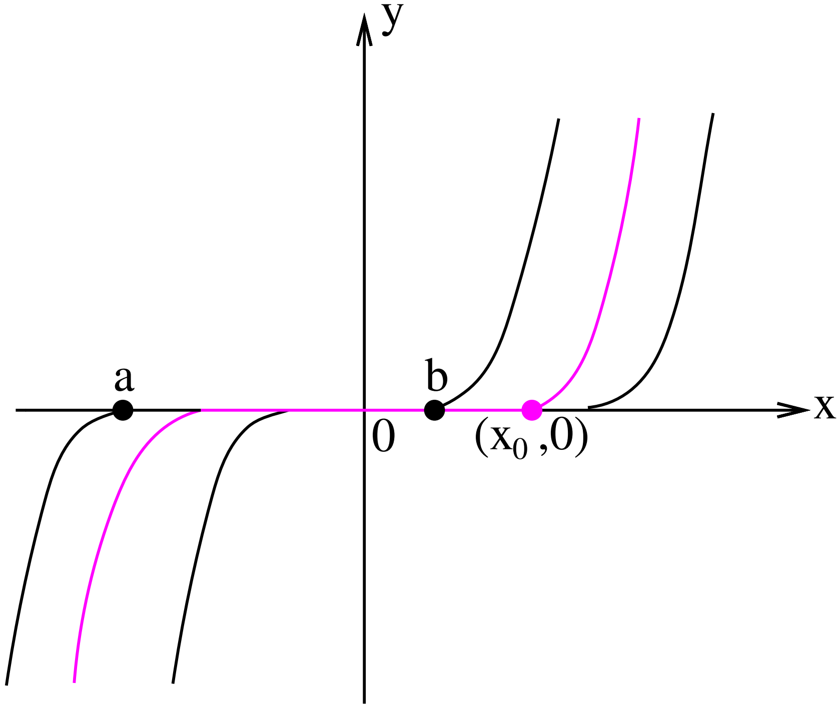


Sl. 2

Postoji još beskonačno mnogo rešenja koja nisu obuhvaćena opštim (Sl. 3):

$$y \equiv 0$$

$$y_{a,b} = \begin{cases} (x-a)^3, & x \leq a \\ 0, & a \leq x \leq b \\ (x-b)^3, & x \geq b \end{cases}$$



Sl. 3

Kroz svaku tačku $(x_0, 0)$ prolazi beskonačno mnogo rešenja.

2. Egzistencija rešenja

$$y' = f(x, y)$$

$f(x, y)$ je definisana na oblasti $D \subset R^2,\ (x_0, y_0) \in D$

Osnovni problem: postoji li rešenje jednačine $y' = f(x, y)$ koje zadovoljava uslov $y(x_0) = y_0$? Ako postoji, da li je jedinstveno?

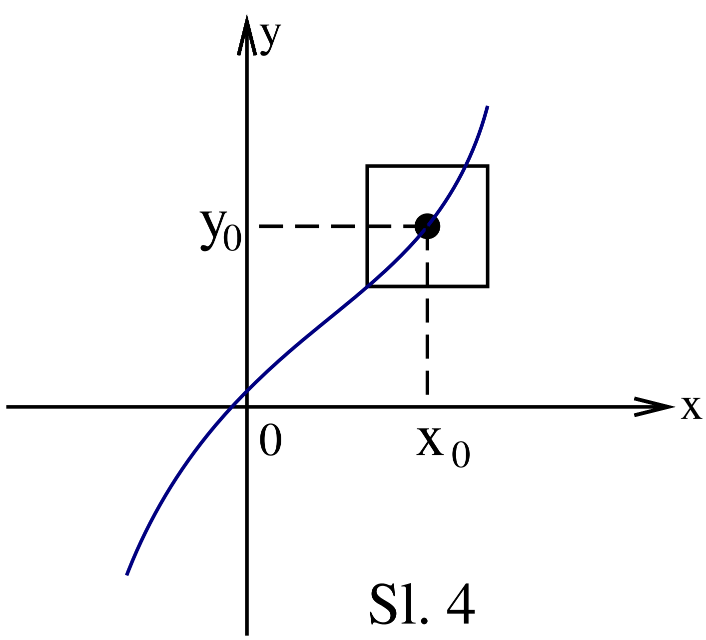
Ovo je tzv. **Košijev problem:**

$$(KP) \qquad \boxed{y' = f(x,y), \quad y(x_0) = y_0}$$

Primer 1. $\quad y' = 3\sqrt[3]{y^2}$

Kroz tačke $(x_0, y_0),\ y_0 \neq 0$ prolazi jedno rešenje

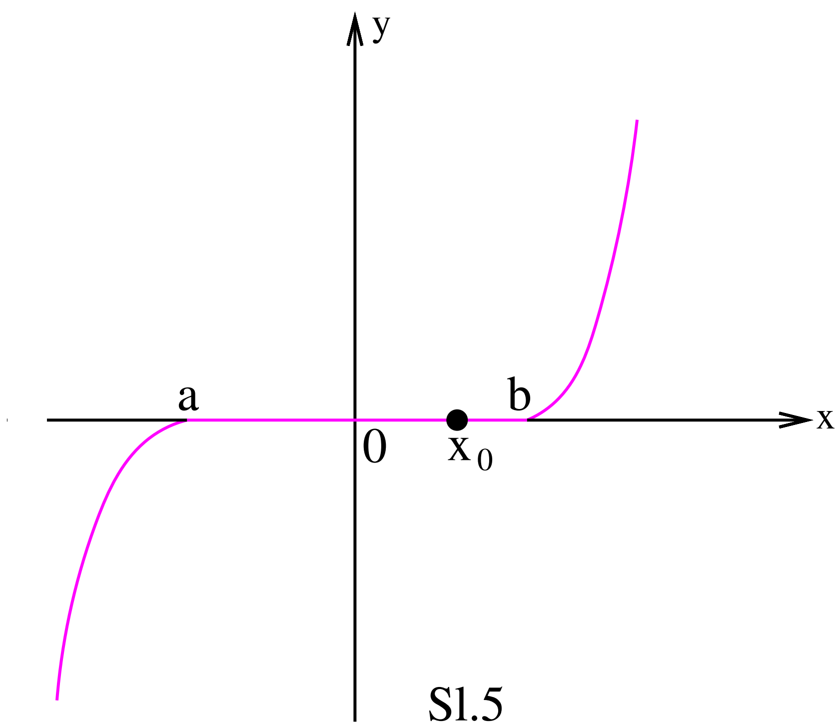
$$y = (x - x_0 + \sqrt[3]{y_0})^3 :$$



Sl. 4

Kroz tačke $(x_0, 0)$ prolazi beskonačno mnogo rešenja (sl.5):

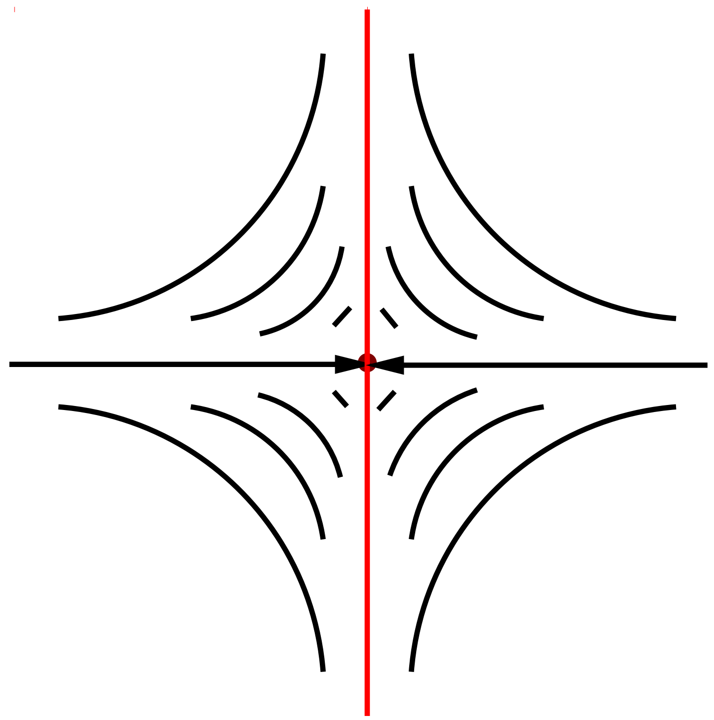
$$y = 0, \quad y_{a,b} = \begin{cases} (x-a)^3, & x \le a \\ 0, & a \le x \le b \\ (x-b)^3, & x \ge b \end{cases}$$



Sl.5

Primer 2. $\quad y' = -\frac{y}{x}$

Može se proveriti da je $y = \dfrac{C}{x}$ opšte rešenje jednačine



Sl. 6

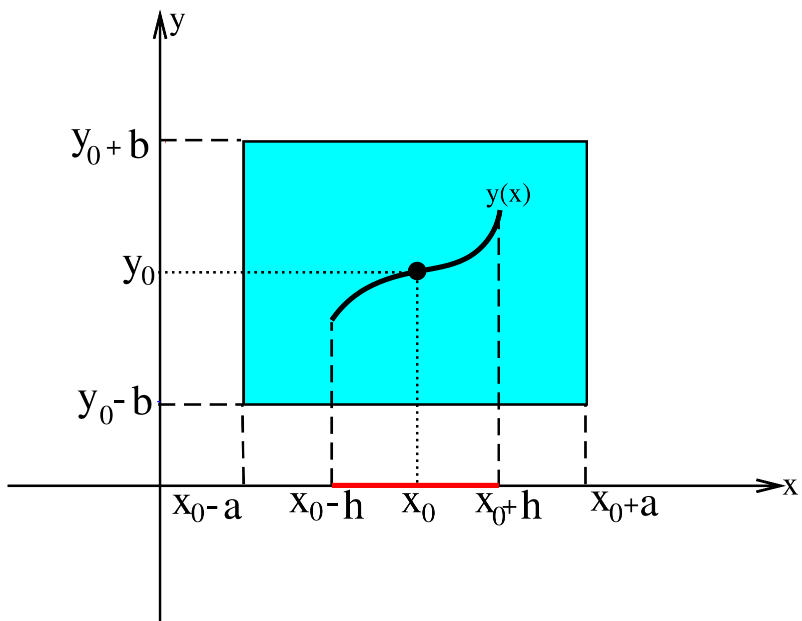
Kroz tačke $(0, y_0)$ ne prolazi ni jedno rešenje

Teorema (Pikar): Neka je $P_{a,b} = \{(x, y) : |x - x_0| \leq a, |y - y_0| \leq b\}$. Ako je $f(x, y)$ na $P_{a,b}$:

- neprekidna
- ograničena, tj. $|f(x, y)| \leq M, \ \forall (x, y) \in P_{a,b}$
- zadovoljava Lipšicov uslov

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|, \ (x, y_1), (x, y_2) \in P_{a,b},$$

tada na $(x_0 - h, x_0 + h),\ h = min\{a, b/M\}$, postoji jedinstveno rešenje (KP) i to rešenje je jedinstveno.



Sl. 7

Napomena: Ako na $P_{a,b}$ postoji $\frac{\partial f}{\partial y}$ i važi $|\frac{\partial f}{\partial y}| \leq L$, tada važi Lipšicov uslov:

$$|f(x, y_1) - f(x, y_2)| = \left|\frac{\partial f}{\partial y}(x, \xi)(y_1 - y_2)\right| \leq L|y_1 - y_2|$$

Primer: $\quad y' = x + y,\ y(0) = 1$

$$P_{a,b} = \{(x, y) : |x| \leq a, |y - 1| \leq b\}$$

Funkcija $f(x,y) = x + y$ na $P_{a,b}$ je:

- neprekidna

- $|f(x,y)| = |x+y| \leq |x|+|y-1|+1 \leq a+b+1 = M$

- $\frac{\partial f}{\partial y} = 1$

Zaključak: Na intervalu $(-h,h)$, $h = min\{a, \frac{b}{a+b+1}\}$, postoji rešenje jednačine i jedinstveno je.

Skica dokaza Pikarove teoreme - metoda sukcesivnih aproksimacija:

$$(KP) \qquad \frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$

$$dy(x) = f(x, y(x))dx$$

$$\int\limits_{x_0}^{x} dy(x) = \int\limits_{x_0}^{x} f(x, y(x))dx$$

$$y(x) - \underbrace{y(x_0)}_{y_0} = \int\limits_{x_0}^{x} f(x, y(x))dx$$

Integralna jednačina:

$$y(x) = y_0 + \int_{x_0}^{x} f(x, y(x))dx$$

Sukcesivne aproksimacije:

$$\begin{aligned}
y_0(x) &\equiv y_0 \\
y_1(x) &= y_0 + \int_{x_0}^{x} f(x, y_0(x))dx \\
&\vdots \\
y_{n+1}(x) &= y_0 + \int_{x_0}^{x} f(x, y_n(x))dx \\
&\vdots
\end{aligned}$$

Rezultat:

$$\boxed{y_n(x) \to y(x), \quad x \in (x_0 - h, x_0 + h)}$$

gde je $y(x)$ *jedinstveno rešenje* (KP)

Ocena greške:

$$|y_n(x) - y(x)| \le ML^n \frac{|x - x_0|^{n+1}}{(n+1)!} e^{L|x-x_0|}$$

Primer: $y' = x + y,\ y(0) = 1$

$$\begin{aligned}
y_0(x) &\equiv 1 \\
y_1(x) &= 1 + \int_0^x f(x, y_0(x))dx = 1 + \int_0^x (x+1)dx \\
&= 1 + \frac{x^2}{2} + x \\
y_2(x) &= 1 + \int_0^x (x + 1 + \frac{x^2}{2} + x)dx = 1 + \frac{x^3}{6} + x^2 + x \\
&\vdots
\end{aligned}$$

Domaći: Ispitati uslove Pikarove teoreme za Košijev problem

$$y' = \frac{x}{y}, \quad y(2) = 4$$

Napomena: Ako je poznato opšte rešenje, rešenje (KP) se dobija zamenom početnih uslova u opštem rešenju:

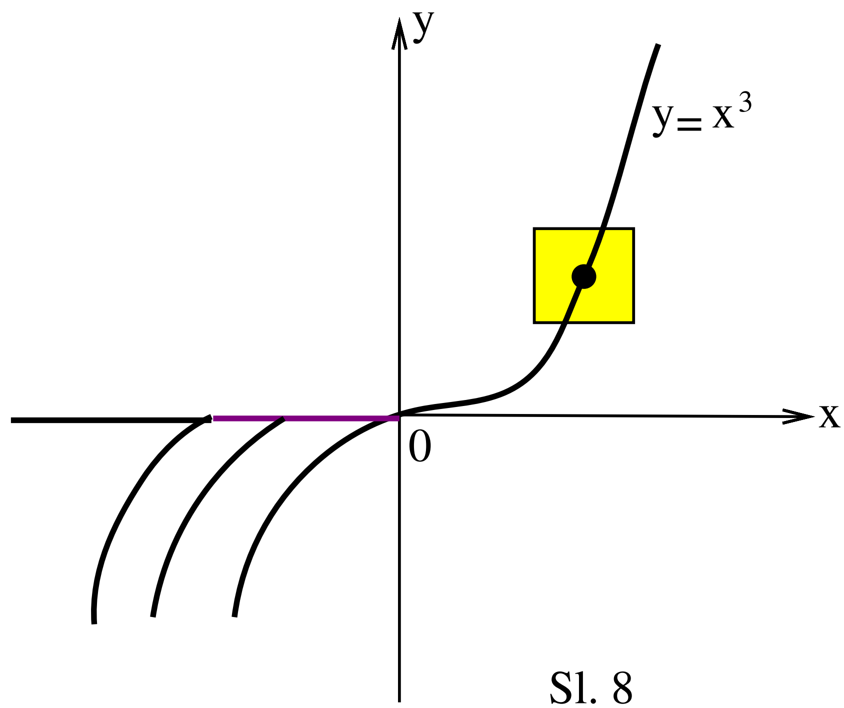
$y = y(x, C)$

$y_0 = y(x_0, C) \Rightarrow C = C_0$

Traženo rešenje (KP): $y = y(x, C_0)$

Primer: $y' = 3\sqrt[3]{y^2},\ y(1) = 1$

$y = (x - C)^3$
$1 = (1 - C)^3 \Rightarrow C = 0$
$y = x^3$



Sl. 8

METODE REŠAVANJA DIFERENCIJALNIH JEDNAČINA PRVOG REDA

$a)$ Jednačina koja razdvaja promenljive

$$\boxed{y' = g(x)h(y)}$$

g, h - neprekidne funkcije

$$\begin{aligned} \frac{dy}{dx} &= g(x)h(y) \qquad h(y) \neq 0 \\ \frac{dy}{h(y)} &= g(x)dx \\ \int \frac{dy}{h(y)} &= \int g(x)dx \quad (OR) \end{aligned}$$

Napomene:

- Ako je $h \equiv 0$, j-na je trivijalna: $y' = 0$
- Ako je $h \not\equiv 0$, ali postoji y_0 t. d. $h(y_0) = 0$, onda je $y = y_0$ rešenje koje se ne sadrži u opštem

Primer: $y' = -\dfrac{x^3}{(y+1)^2} \Rightarrow \begin{array}{rcl} g(x) & = & -x^3 \\ h(y) & = & \dfrac{1}{(y+1)^2} \end{array}$

$$\begin{array}{rcl} (y+1)^2 dy & = & -x^3 dx \\ \displaystyle\int (y+1)^2 dy & = & -\displaystyle\int x^3 dx \Rightarrow \end{array}$$

$\frac{(y+1)^3}{3} = -\frac{x^4}{4} + C$ implicitno zadato rešenje

$y = \sqrt[3]{3\left(C - \frac{x^4}{4}\right)} - 1$ eksplicitno zadato rešenje

$(KP) \quad y(0) = 1 : \qquad 1 = \sqrt[3]{3C} - 1 \Rightarrow C = 8/3$

$$y = \sqrt[3]{3(8/3 - x^4/4)} - 1$$

b) Homogena jednačina

$$\boxed{y' = g\left(\frac{y}{x}\right)}$$

$x \neq 0,\ g -$ neprekidna funkcija

Smena: $z = \frac{y}{x} \Rightarrow y = zx \Rightarrow y' = z'x + z$

$$\begin{aligned} z'x + z &= g(z) \\ z'x &= g(z) - z \\ \frac{dz}{g(z) - z} &= \frac{dx}{x}, \quad g(z) - z \neq 0 \end{aligned}$$

$\int \frac{dz}{g(z)-z} = \ln|x| + C$ opšte rešenje

Primer: $y' = \frac{y^2+xy-x^2}{x^2} = \left(\frac{y}{x}\right)^2 + \frac{y}{x} - 1$

$\frac{y}{x} = z \Rightarrow z'x + z = z^2 + z - 1 \Rightarrow$

$$\begin{aligned} z'x &= z^2 - 1 \\ \frac{dz}{z^2 - 1} &= \frac{dx}{x} \\ \frac{1}{2}\ln\left|\frac{z-1}{z+1}\right| &= \ln|x| + C \\ \frac{1}{2}\ln\left|\frac{y-x}{y+x}\right| &= \ln|x| + C \end{aligned}$$

$c)$ Jednačina koja se svodi na homogenu

$$\boxed{y' = F\left(\frac{\alpha x+\beta y+\gamma}{ax+by+c}\right)}$$

$F-$ neprekidna; $\gamma \neq 0$ ili $c \neq 0$

$\mathbf{1}^\circ \quad \begin{vmatrix} \alpha & \beta \\ a & b \end{vmatrix} = \alpha b - \beta a \neq 0$

Smene:

$$\begin{aligned} x &= X + h \qquad X\text{- nova nezavisno promenljiva} \\ y &= Y + k \qquad Y\text{- nova nepoznata funkcija} \end{aligned}$$

$$\frac{dy}{dx} = \frac{dY}{dX} = F\left(\frac{\alpha(X+h)+\beta(Y+k)+\gamma}{a(X+h)+b(Y+k)+c}\right)$$

$$\begin{aligned} \alpha h + \beta k + \gamma &= 0 \\ ah + bk + c &= 0, \quad (det \neq 0) \end{aligned}$$

$$\frac{dY}{dX} = F\left(\frac{\alpha X + \beta Y}{aX + bY}\right) = F\left(\frac{\alpha + \beta\frac{Y}{X}}{a + b\frac{Y}{X}}\right)$$

$\mathbf{2^\circ}$ $\quad \alpha b - \beta a = 0$

Smena: $z = \alpha x + \beta y$ $\qquad$ (ili $z = ax + by$)

Jednačina se svodi na j-nu koja razdvaja promenljive u odnosu na novu nepoznatu f-ju z

Domaći (jun 2005): $y' = \frac{(2y+1)^2}{(x+2y-2)^2}$

d) **Linearna diferencijalna jednačina prvog reda**

$$
\begin{aligned}
(LNH) \qquad y' + p(x)y &= q(x) \\
(LH) \qquad y' + p(x)y &= 0
\end{aligned}
$$

p, q - neprekidne funkcije

$$
\begin{aligned}
\frac{dy}{dx} &= -p(x)y, \qquad (y \neq 0) \\
\frac{dy}{y} &= -p(x)dx \\
\ln|y| &= -\int p(x)dx + C_1 \\
|y| &= e^{-\int p(x)dx} e^{C_1} \qquad (e^{C_1} > 0)
\end{aligned}
$$

$$C = \begin{cases} e^{C_1} & \text{za } y > 0 \\ -e^{C_1} & \text{za } y < 0 \end{cases}$$

$$y = Ce^{-\int p(x)dx} \qquad (C \neq 0)$$

$$y \equiv 0 \qquad \text{rešenje}$$

$\boxed{y_h = Ce^{-\int p(x)dx}}$, $C \in R$ - opšte rešenje (LH)

Metoda varijacije konstanata za (LNH) jednačinu:

$$\begin{aligned} y_{nh} &= C(x)e^{-\int p(x)dx} \\ y'_{nh} &= C'e^{-\int p(x)dx} - Ce^{-\int p(x)dx}p(x) \end{aligned}$$

(LNH):

$$\underbrace{C'e^{-\int p(x)dx} - Ce^{-\int p(x)dx}p(x)}_{y'_{nh}} + \underbrace{p(x)Ce^{-\int p(x)dx}}_{y_{nh}} = q(x)$$

$$\begin{aligned} C'(x) &= e^{\int p(x)dx}q(x) \\ C(x) &= \int q(x)e^{\int p(x)dx}dx + C \end{aligned}$$

$$\boxed{y_{nh} = e^{-\int p(x)dx}\left(\int q(x)e^{\int p(x)dx}dx + C\right)}$$

Primer: $y' + 2xy = 4x$

$$(LH): \qquad y' + 2xy = 0$$
$$\frac{dy}{y} = -2xdx \Rightarrow \ln|y| = -x^2 + C_1$$

$$\begin{aligned} y_h &= Ce^{-x^2} \\ y_{nh} &= C(x)e^{-x^2} \end{aligned}$$

$$C'e^{-x^2} - 2xe^{-x^2}C + 2xCe^{-x^2} = 4x$$
$$C' = 4xe^{x^2} \ \Rightarrow \ C(x) = 2e^{x^2} + C$$

$$y_{nh} = e^{-x^2}\left(2e^{x^2} + C\right) = \underbrace{Ce^{-x^2}}_{y_h} + \underbrace{2}_{y_p}$$

Važi u opštem slučaju:

$$y_{nh} = \underbrace{e^{-\int p(x)dx}\int q(x)e^{\int p(x)dx}dx}_{y_p} + \underbrace{Ce^{-\int p(x)dx}}_{y_h}$$

e) Bernulijeva jednačina

$$\boxed{y' + p(x)y = q(x)y^{\alpha}}$$

$\alpha = 0$: linearna

$\alpha = 1$: razdvaja promenljive

$$y'y^{-\alpha} + p(x)y^{1-\alpha} = q(x)$$

Smena: $z = y^{1-\alpha}$

$$z' = (1-\alpha)y^{-\alpha}y' \Rightarrow y'y^{-\alpha} = \frac{z'}{1-\alpha}$$

$$\frac{z'}{1-\alpha} + p(x)z = q(x) \ \mid \ \cdot (1-\alpha)$$

$$z' + p_1(x)z = q_1(x), \quad \text{linearna}$$

$p_1 = (1-\alpha)p,\ q_1 = (1-\alpha)q$

Napomena: Ako je y_1 jedno rešenje **Rikatijeve jednačine**

$$y' + p(x)y^2 + q(x)y = r(x),$$

smena $y = y_1 + z$ svodi jednačinu na Bernulijevu

f) Jednačina sa totalnim diferencijalom

- Ako $F(x,y)$ ima neprekidne parcijalne izvode na $D \subseteq R^2$,onda je

$$dF(x,y) = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$$

- Obratno: ako su $P(x,y)$ i $Q(x,y)$ date funkcije, postoji li $F(x,y)$ tako da je

$$dF(x,y) = P(x,y)dx + Q(x,y)dy?$$

Teorema: Neka su $P(x,y)$ i $Q(x,y)$ neprekidne zajedno sa parcijalnim izvodima na D. Tada je izraz $P(x,y)dx+Q(x,y)dy$ totalni diferencijal neke funkcije $F(x,y)$ **ako i samo ako** je

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{na } D$$

Dokaz. $(\Rightarrow)$: $Pdx + Qdy$ je tot. dif. $\Rightarrow$

$$\exists F \text{ t.d. } \quad \frac{\partial F}{\partial x} = P,\ \frac{\partial F}{\partial y} = Q \ \Rightarrow$$

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial P}{\partial y},\ \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial Q}{\partial x} \quad \Rightarrow \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$(\Leftarrow)$: $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Tražimo F t.d.

$$(1) \quad \frac{\partial F}{\partial x} = P(x,y) \qquad (2) \quad \frac{\partial F}{\partial y} = Q(x,y)$$

$$(1) \ \Rightarrow \ F(x,y) = \int_a^x P(x,y)dx + \varphi(y)$$

$$\begin{aligned} \frac{\partial F}{\partial y} &= \int_a^x \frac{\partial P}{\partial y} dx + \varphi'(y) = \int_a^x \frac{\partial Q}{\partial x} dx + \varphi'(y) \\ &= Q(x,y) - Q(a,y) + \varphi'(y) = Q(x,y) \end{aligned}$$

$$\Rightarrow \varphi'(y) = Q(a,y) \;\Rightarrow\; \varphi(y) = \int\limits_b^y Q(a,y)dy + C_1$$

$$\boxed{F(x,y) = \int\limits_a^x P(x,y)dx + \int\limits_b^y Q(a,y)dy + C_1}$$

Primena na diferencijalne jednačine:

$$y' = f(x,y) = -\frac{P(x,y)}{Q(x,y)}$$

$$\frac{dy}{dx} = -\frac{P(x,y)}{Q(x,y)} \quad \Rightarrow$$

(3) $$\boxed{P(x,y)dx + Q(x,y)dy = 0}$$

Ako je $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, jednačina (3) je **jednačina sa totalnim diferencijalom** i može se zapisati u obliku

(4) $$dF(x,y) = 0$$

Opšte rešenje jednačine (4) : $F(x,y) = C_2$, tj.

$$\boxed{\int\limits_a^x P(x,y)dx + \int\limits_b^y Q(a,y)dy = C}$$

Primer: $\underbrace{(3x^2 + 6xy^2)}_{P}dx + \underbrace{(6x^2y + 4y^2)}_{Q}dy = 0$

$$\frac{\partial P}{\partial y} = 12xy = \frac{\partial Q}{\partial x}$$

$$\begin{aligned} F(x,y) &= \int P(x,y)dx + \varphi(y) = \int (3x^2 + 6xy^2)dx + \varphi(y) \\ &= x^3 + 3x^2y^2 + \varphi(y) \end{aligned}$$

$$\frac{\partial F}{\partial y} = Q(x,y): \ 6x^2y + \varphi'(y) = 6x^2y + 4y^2 \ \Rightarrow \ \varphi'(y) = 4y^2$$

$$\varphi(y) = \frac{4}{3}y^3 + C_1 \ \Rightarrow \ F(x,y) = x^3 + 3x^2y^2 + \frac{4}{3}y^3 + C_1$$

Opšte rešenje: $x^3 + 3x^2y^2 + \frac{4}{3}y^3 = C$

Integracioni faktor:

$$P(x,y)dx + Q(x,y)dy = 0, \ | \cdot r(x,y)$$

sa $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$

$$P_1(x,y)dx + Q_1(x,y)dy = 0,$$

$P_1 = P \cdot r, \ Q_1 = Q \cdot r$

$r(x,y)$ je **integracioni faktor** ako je $\frac{\partial P_1}{\partial y} = \frac{\partial Q_1}{\partial x}$

Ako je:

$1^\circ \ \dfrac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q(x,y)} = f(x) \ \Rightarrow \ r = e^{\int f(x)dx}$ je int. faktor

$2^\circ \ \dfrac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{P(x,y)} = g(y) \ \Rightarrow \ r = e^{-\int g(y)dy}$ je int. faktor

Dokaz 1° : $\quad P_1 = Pe^{\int f(x)dx}, \ \ Q_1 = Qe^{\int f(x)dx}$

$$
\begin{aligned}
\frac{\partial P_1}{\partial y} &= \frac{\partial P}{\partial y} e^{\int f(x)dx} \\
\frac{\partial Q_1}{\partial x} &= \frac{\partial Q}{\partial x} e^{\int f(x)dx} + Q e^{\int f(x)dx} f(x) = \\
&= \frac{\partial Q}{\partial x} e^{\int f(x)dx} + Q e^{\int f(x)dx} \frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q} = \\
&= \frac{\partial P}{\partial y} e^{\int f(x)dx} = \frac{\partial P_1}{\partial y}
\end{aligned}
$$

Domaći (januar 05): Rešiti jednačinu

$$2y' \tan x + \frac{\sin 2y}{\cos^2 x} = 0$$

nalaženjem integracionog faktora $r(y)$

4. Singularne tačke

Tačke u kojima su za jednačine

$$\frac{dy}{dx} = f(x, y) \quad \text{i} \quad \frac{dx}{dy} = \frac{1}{f(x, y)}$$

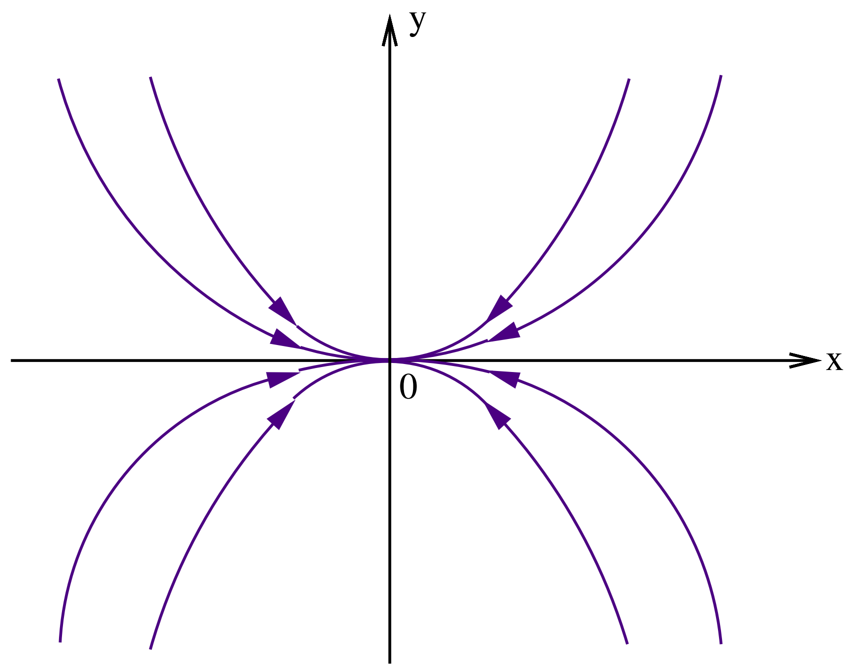
narušeni uslovi Pikarove teoreme

Primer: $\frac{dy}{dx} = \frac{Ax+By}{Cx+Dy}$ $(0,0)$ je sing. tačka

Tipovi singulariteta (kroz primere):

1° $\frac{dy}{dx} = \frac{2y}{x}$ $\left(\frac{dx}{dy} = \frac{x}{2y}\right)$

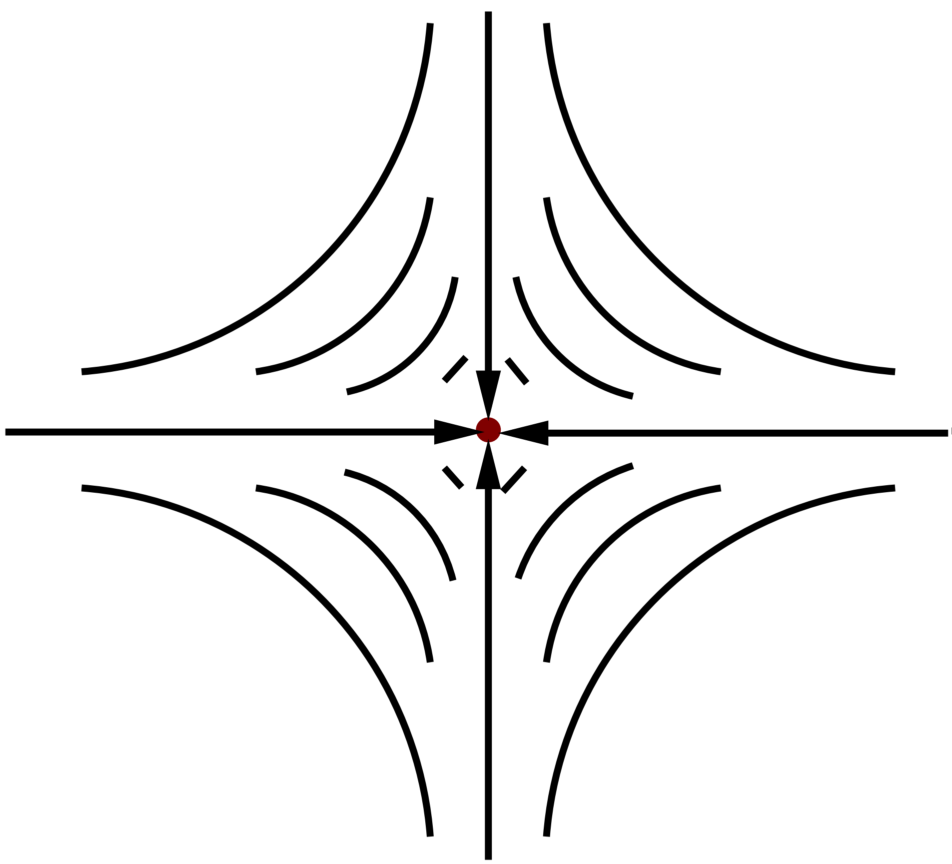
Opšte rešenje: $y = Cx^2,\ y \equiv 0 \quad (x \equiv 0)$



čvor - kroz $(0,0)$ prolaze sve integralne krive

$2^\circ \ \frac{dy}{dx} = -\frac{y}{x} \quad \left(\frac{dx}{dy} = -\frac{x}{y}\right)$

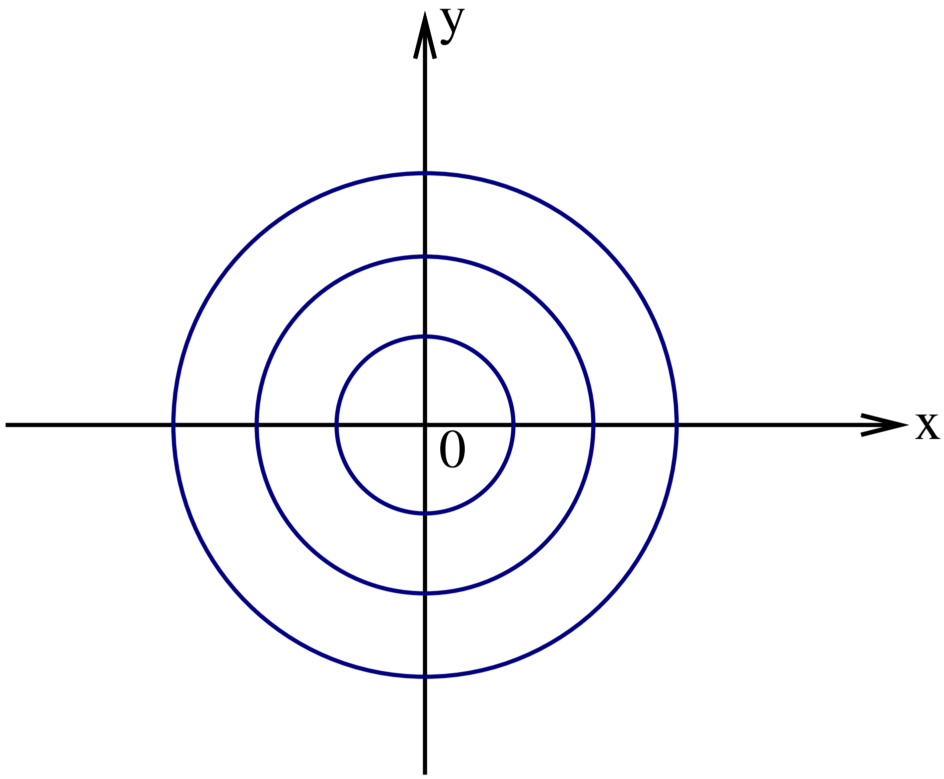
Opšte rešenje: $y = \frac{C}{x},\ y \equiv 0 \quad (x \equiv 0)$



sedlo - kroz $(0,0)$ ne prolazi ni jedna integralna kriva (osim degenerisanih)

3° $\frac{dy}{dx} = -\frac{x}{y} \quad \left(\frac{dx}{dy} = -\frac{y}{x}\right)$

Opšte rešenje: $x^2 + y^2 = C$

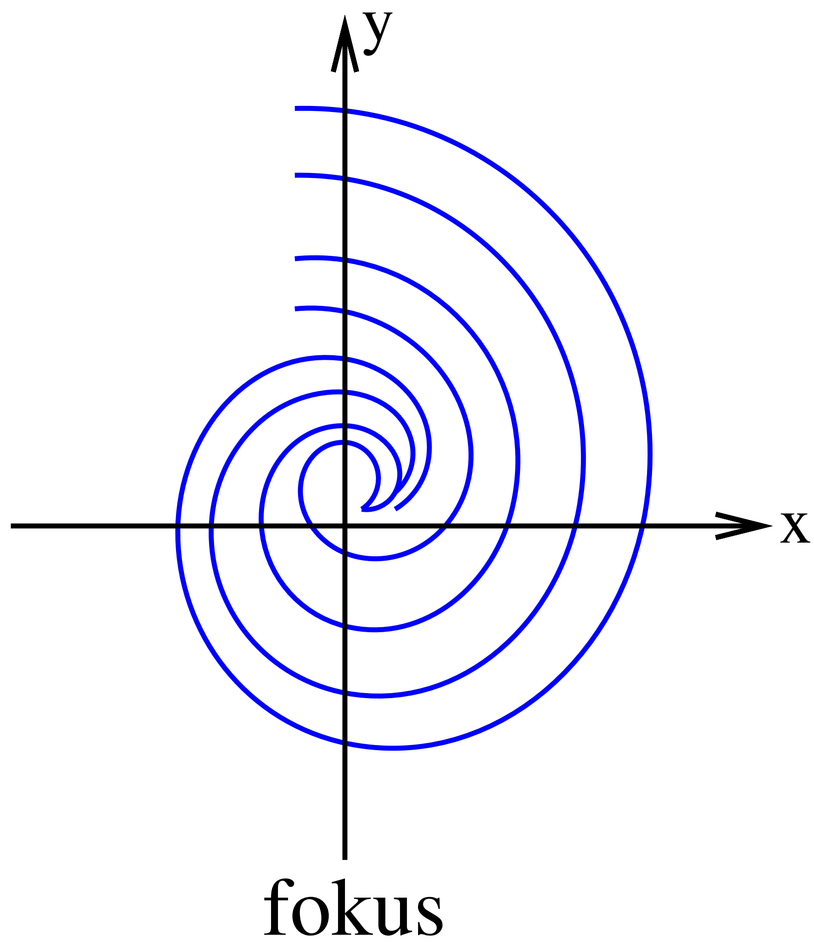


centar - kroz $(0, 0)$ ne prolazi ni jedna integralna kriva

4° $\frac{dy}{dx} = \frac{x+y}{x-y} \quad \left(\frac{dx}{dy} = \frac{x-y}{x+y}\right)$

Opšte rešenje: $\sqrt{x^2+y^2} = Ce^{\operatorname{arctan}\frac{y}{x}} \quad (C > 0)$

$$x = \rho\cos\varphi,\ y = \rho\sin\varphi \quad \Rightarrow \quad \rho = Ce^{\varphi}$$



fokus

DIFERENCIJALNE JEDNAČINE VIŠEG REDA

1. Osnovni pojmovi. Egzistencija rešenja

Opšti oblik diferencijalne j-ne $n-$tog reda:

(1) $$F(x, y, y', \ldots, y^{(n)}) = 0$$

Normalni oblik:

(2) $$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$$

Definicija: Funkcija $y = y(x)$ je **rešenje** jednačine (2) na (a, b) ako identički zadovoljava jednačinu (2) na (a, b)

Definicija: Funkcija

$$y = y(x, C_1, \ldots, C_n) \quad (\varphi(x, y, C_1, \ldots, C_n) = 0)$$

je **opšte rešenje** j-ne (2) na (a, b) ako identički zadovoljava (2) po x i po $C_1, \ldots, C_n$ na (a, b)

Košijev problem:

$$
\begin{array}{lrcl}
(KP) & y^{(n)} & = & f(x, y, y', \ldots, y^{(n-1)}) \\
 & y(x_0) & = & y_0 \\
 & y'(x_0) & = & y'_0 \\
 & \vdots & & \\
 & y^{(n-1)}(x_0) & = & y_0^{(n-1)}
\end{array}
$$

Pitanje: Da li postoji rešenje (KP)? Ako postoji, da li je jedinstveno?

Pikarova teorema: Neka je

$$
P_{a,b} = \{(x, y, y', \ldots, y^{(n-1)}) : \quad |x - x_0| \le a, |y - y_0| \le b, \ldots, |y^{(n-1)} - y_0^{(n-1)}| \le b\}
$$

Ako je

1° f neprekidna na $P_{a,b}$

2° $|f(x, y, \ldots, y^{(n-1)})| \le M$ na $P_{a,b}$

3° $|f(x, y, y', \ldots, y^{(n-1)}) - f(x, \bar{y}, \bar{y}', \ldots, \bar{y}^{(n-1)})| \le$
$\le L(|y - \bar{y}| + \cdots + |y^{(n-1)} - \bar{y}^{(n-1)})|$, (Lipšicov uslov)

tada na $(x_0 - h, x_0 + h)$, $h = min\{a, b/M\}$, postoji rešenje $y = y(x)$ (KP) i jedinstveno je.

Napomena:

$$\frac{\partial f}{\partial y}, \frac{\partial f}{\partial y'}, \ldots, \frac{\partial f}{\partial y^{(n-1)}} \text{ ograničeni } \Rightarrow \text{ Lipšicov uslov}$$

2. Snižavanje reda

1° $\boxed{F(x, y', y'') = 0}$ **ne sadrži** y

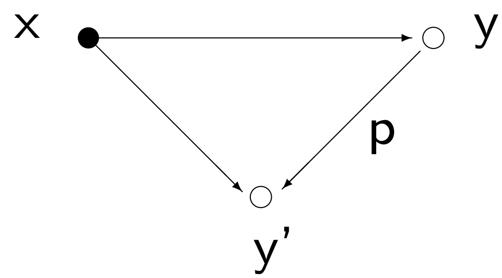
Smena: $y' = z \;\Rightarrow y'' = z'$

$F(x, z, z') = 0$ j-na prvog reda

$z = z(x, C_1)$ opšte rešenje

$y = \int z(x, C_1)dx + C_2$ opšte rešenje polazne j-ne

2° $\boxed{F(y, y', y'') = 0}$ **ne sadrži** x



Smena: $\quad y' = p(y)$
p - nova nepoznata funkcija koja zavisi od y

$y'' = \frac{d}{dx}\underbrace{p(y)}_{y'} = \frac{dp}{dy} \cdot \frac{dy}{dx} = \frac{dp}{dy}p$

$$F(y, p, \frac{dp}{dy}p) = 0 \qquad \text{dif. j-na prvog reda}$$

$p = p(y, C_1)$ opšte rešenje $\Rightarrow \frac{dy}{dx} = p(y, C_1) \Rightarrow$

$\frac{dy}{p(y,C_1)} = dx \Rightarrow \int \frac{dy}{p(y,C_1)} = x + C_2$ o.r. polazne j-ne

3. Linearna diferencijalna jednačina drugog reda

a) Definicija, egzistencija rešenja

$$\begin{array}{rrcl} (LNH) & y'' + p(x)y' + q(x)y & = & f(x) \\ (LH) & y'' + p(x)y' + q(x)y & = & 0 \end{array}$$

(KP) za (LNH) ili (LH) :

$$y(x_0) = y_0,\ y'(x_0) = y_0'$$

Teorema (Pikarova, globalni iskaz): Ako su $p(x),\ q(x)$ i $f(x)$ neprekidne na (a, b) i ako $x_0 \in (a, b)$, tada postoji jedno i samo jedno rešenje (KP) definisano na (a, b)

Napomena: moguće je $a = -\infty,\ b = +\infty$

Primer: $y'' + \frac{1}{x} y' + \frac{1}{5-x} y = \ln x;\quad y(1) = 0,\ y'(1) = 1$

$$p(x) = \frac{1}{x},\ q(x) = \frac{1}{5-x},\ f(x) = \ln x$$

p je nepr. za $x \neq 0$, $\quad q$ je nepr. za $x \neq 5$
f je nepr. za $x > 0$

Na $(a, b) = (0, 5)$ je definisano rešenje (KP) i jedinstveno je.

Napomena: Teorema važi i za $n = 1$:

$$y' + p(x)y = q(x)$$

b) **Osobine rešenja linearne homogene jednačine**

$$(LH) \qquad y'' + p(x)y' + q(x)y = 0$$

Napomena: $y \equiv 0$ je rešenje, tzv. trivijalno

Teorema: Ako su $y_1(x)$ i $y_2(x)$ rešenja (LH), a C_1 i C_2 proizvoljne konstante, onda je

$$y = C_1 y_1(x) + C_2 y_2(x)$$

rešenje (LH)

Dokaz: $y' = C_1 y_1' + C_2 y_2', \quad y'' = C_1 y_1'' + C_2 y_2'' \ \rightarrow$ (LH) :

$$C_1 y_1'' + C_2 y_2'' + p(x)(C_1 y_1' + C_2 y_2') + q(x)(C_1 y_1 + C_2 y_2) =$$
$$= C_1 \underbrace{(y_1'' + p(x)y_1' + q(x)y_1)}_{0} + C_2 \underbrace{(y_2'' + p(x)y_2' + q(x)y_2)}_{0}$$
$$\equiv 0$$

Napomena : $y = C_1 y_1(x) + C_2 y_2(x)$ je opšte rešenje (LH) ('**slabo**')

Definicija: Rešenja $y_1(x)$ i $y_2(x)$ su **linearno zavisna** na (a, b) ako je

$$\frac{y_2(x)}{y_1(x)} \equiv \text{ const}, \ x \in (a, b),$$

a **linearno nezavisna** na (a, b) ako je

$$\frac{y_2(x)}{y_1(x)} \equiv u(x), \ x \in (a, b)$$

Napomena:

$$\frac{y_2}{y_1} = \text{ const} \Leftrightarrow C_1 y_1 + C_2 y_2 \equiv 0 \ (C_1 \neq 0 \text{ ili } C_2 \neq 0)$$

$$\frac{y_2}{y_1} = u(x) \Leftrightarrow C_1 y_1 + C_2 y_2 \neq 0, \ \forall x \in (a, b)$$

(osim za $C_1 = C_2 = 0$)

Definicije (alternativne): Rešenja $y_1(x)$ i $y_2(x)$ su **linearno zavisna** na (a, b) ako postoje konstante C_1 i C_2, koje nisu obe nule , takve da je

$$C_1 y_1(x) + C_2 y_2(x) = 0, \quad x \in (a, b)$$

Rešenja $y_1(x)$ i $y_2(x)$ su **linearno nezavisna** na (a, b) ako važi implikacija

$$C_1 y_1(x) + C_2 y_2(x) = 0, \ x \in (a, b) \ \Rightarrow \ C_1 = 0, C_2 = 0$$

Primer: $y_1(x) = x$ i $y_2(x) = x+1$ su rešenja jednačine $y'' = 0$

$$C_1 x + C_2(x+1) = (C_1 + C_2)x + C_2 \equiv 0 \;\Rightarrow$$

$C_1 + C_2 = 0, C_2 = 0 \;\Rightarrow C_1 = 0, C_2 = 0$ (lin. nez. reš.)

$\left(\frac{y_2(x)}{y_1(x)} \;=\; \frac{x+1}{x} \;= u(x) \right)$

Definicija: Neka su $y_1(x)$ i $y_2(x)$ rešenja (LH) jednačine. **Vronskijan** rešenja $y_1(x)$ i $y_2(x)$ je determinanta

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

Teorema: Rešenja $y_1(x)$ i $y_2(x)$ (LH) jednačine su linearno nezavisna na (a,b) **ako i samo ako** je

$$W(x) \neq 0, \quad \forall x \in (a,b)$$

Dokaz: $(\Rightarrow)$ Neka su $y_1(x)$ i $y_2(x)$ lin. nezavisna na (a,b). Pretpostavimo, suprotno tvrdnji teoreme, da

$$\exists x_0 \in (a,b): \quad W(x_0) = 0$$

Posmatramo homogeni sistem (S) linearnih algebarskih jednačina po $C_1,\ C_2$

$$\begin{aligned} C_1 y_1(x_0) + C_2 y_2(x_0) &= 0 \\ C_1 y_1'(x_0) + C_2 y_2'(x_0) &= 0. \end{aligned}$$

Zbog $det(S) = W(x_0) = 0$, imaće netrivijalno rešenje $\overline{C}_1,\ \overline{C}_2$. Tada je

$$y = \overline{C}_1\, y_1(x) + \overline{C}_2\, y_2(x)$$

rešenje j-ne koje zadovoljava početne uslove

$$\begin{aligned} y(x_0) &= \overline{C}_1 y_1(x_0) + \overline{C}_2 y_2(x_0) = 0 \qquad (S) \\ y'(x_0) &= \overline{C}_1 y_1'(x_0) + \overline{C}_2 y_2'(x_0) = 0 \qquad (S) \end{aligned}$$

Iste početne uslove zadovoljava i rešenje $y \equiv 0$ $\Rightarrow$(Pikarova teorema)

$$C_1\, y_1(x) + \overline{C}_2\, y_2(x) \equiv 0, \quad x \in (a, b)$$

Ovo je kontradiktorno pretpostavci o linearnoj nezavisnosti rešenja y_1 i y_2. Pretpostavka $W(x_0) = 0$ se mora odbaciti, pa je $W(x) \neq 0,\ \forall x \in (a, b)$

$(\Leftarrow) \quad W(x) \neq 0,\ \forall x \in (a,b)$

Pretpostavimo $C_1 y_1(x) + C_2 y_2(x) = 0,\ x \in (a,b)$. Sledi

$$C_1 y_1'(x) + C_2 y_2'(x) = 0,\ x \in (a,b)$$

Za fiksirano $x = x_0 \in (a,b)$:

$$\begin{aligned} C_1 y_1(x_0) + C_2 y_2(x_0) &= 0 \\ C_1 y_1'(x_0) + C_2 y_2'(x_0) &= 0 \end{aligned}$$

Homogen sistem, $det = W(x_0) \neq 0 \Rightarrow C_1 = 0$, $C_2 = 0$, tj. y_1 i y_2 su linearno nezavisni

Primer: $y_1 = x$, $y_2 = x + 1$

$$y'' = 0, \quad (a,b) = (-\infty, \infty)$$

$$W(x) = \begin{vmatrix} x & x+1 \\ 1 & 1 \end{vmatrix} = x-(x+1) = -1 \neq 0,\ x \in (-\infty,\infty)$$

Teorema: Neka su $y_1(x)$ i $y_2(x)$ **linearno nezavisna** rešenja (LH) jednačine. Tada opšte rešenje (LH) jednačine

$$y_h = C_1 y_1(x) + C_2 y_2(x) \qquad (\textbf{'jako'} \text{ rešenje})$$

sadrži sva rešenja (LH) jednačine.

Dokaz: Neka je $y(x)$ proizvoljno rešenje (LH) jednačine koje zadovoljava početne uslove $y(x_0) = y_0$, $y'(x_0) = y'_0$. Pokazaćemo da se mogu naći konstante C_1 i C_2 takve da i $C_1 y_1(x) + C_2 y_2(x)$ zadovolji iste te početne uslove, tj. da bude

$$\begin{aligned} C_1 y_1(x_0) + C_2 y_2(x_0) &= y_0 \\ C_1 y'_1(x_0) + C_2 y'_2(x_0) &= y'_0 \end{aligned}$$

Nehomogen sistem, $det = W(x_0) \neq 0 \Rightarrow$ jedinstveno rešenje: $C_1 = C_1^\circ, C_2 = C_2^\circ$.

Sada $C_1^\circ y_1(x) + C_2^\circ y_2(x)$ i $y(x)$ zadovoljavaju iste početne uslove $\Rightarrow$ (Pikarova t.)

$$C_1^\circ y_1(x) + C_2^\circ y_2(x) \equiv y(x), \ x \in (a, b)$$

c) Rešavanje (LH) jednačine ako je poznato jedno rešenje

$$(LH) \qquad y'' + p(x)y' + q(x)y = 0$$

Teorema: Ako je $y_1(x)$ netrivijalno partikularno rešenje (LH) jednačine,tada se linearno nezavisno rešenje $y_2(x)$ može dobiti rešavanjem (DJ) prvog reda.

Dokaz:
$$\begin{aligned} y_2(x) &= u(x)y_1(x) \\ y_2' &= u'y_1 + uy_1' \\ y_2'' &= u''y_1 + 2u'y_1' + uy_1'' \quad \to (LH) \end{aligned}$$

$$u''y_1 + 2u'y_1' + uy_1'' + p(x)(u'y_1 + uy_1') + q(x)uy_1 = 0$$
$$u'' + 2u'y_1' + p(x)u'y_1 + u(\underbrace{y_1'' + py_1' + qy_1}_{0}) = 0$$
$$u''y_1 + u'(2y_1' + p(x)y_1) = 0$$

Smena: $\quad z = u' \;\Rightarrow\; z' = u''$

$$z'y_1 + z(2y_1' + p(x)y_1) = 0 \qquad \text{j-na prvog reda}$$

$$z' = -z\frac{2y_1' + p(x)y_1}{y_1} \qquad \text{razdvaja prom.}$$

$z = z(x)$ jedno partikularno rešenje $\Rightarrow$
$u = \int z(x)dx$ jedna primitivna funkcija $\Rightarrow$

$y_2(x) = u(x)y_1(x)$

$$y_h = C_1 y_1(x) + C_2 y_2(x)$$

d) Linearna homogena jednačina drugog reda sa konstantnim koeficijentima

$$(LHC) \qquad \boxed{y'' + py' + qy = 0}$$

$p, q \equiv$ const; $(a, b) = (-\infty, \infty)$

Rešenje tražimo u obliku
$$\begin{aligned} y &= e^{\lambda x} \Rightarrow \\ y' &= \lambda e^{\lambda x} \\ y'' &= \lambda^2 e^{\lambda x} \end{aligned}$$

$$\lambda^2 e^{\lambda x} + p\lambda e^{\lambda x} + q e^{\lambda x} = 0 \Rightarrow$$

$$(KJ) \qquad \boxed{\lambda^2 + p\lambda + q = 0}$$ **karakteristična jednačina**

$\lambda_1, \ \lambda_2$ koreni (KJ) :

$1^\circ \quad \lambda_1 \neq \lambda_2$ realni

$2^\circ \quad \lambda_1 = \lambda_2 = a$ realno

$3^\circ \quad \lambda_1, 2 = \alpha \pm i\beta$

1° $y_1(x) = e^{\lambda_1 x},\ y_2(x) = e^{\lambda_2 x} \Rightarrow \frac{y_2}{y_1} = e^{(\lambda_2 - \lambda_1)x} \neq$ const

$$y_h = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

2° $y_1 = e^{ax}, \quad y_2 = u(x) y_1(x) \ \Rightarrow \ c)$

$$u'' y_1 + u'(2y_1' + p y_1) = 0$$

Medjutim,
$\lambda^2 + p\lambda + q = (\lambda - a)^2 = \lambda^2 - 2a\lambda + a^2 \ \Rightarrow \ p = -2a,\ q = a^2$

$$y_1 = e^{ax},\ y_1' = a e^{ax},$$

pa je

$$u'' e^{ax} + u'(2ae^{ax} \underbrace{-2a}_{p} e^{ax}) = 0 \ \Rightarrow \ u'' = 0 \ \Rightarrow$$

$u' = A \ \Rightarrow \ u = Ax + B.$

Spec. $A = 1,\ B = 0 \Rightarrow u = x \Rightarrow y_2(x) = x y_1(x) = x e^{ax} \Rightarrow$

$$y_h = C_1 e^{ax} + C_2 x e^{ax}$$

3° $\lambda_1 = \alpha + i\beta,\ \lambda_2 = \alpha - i\beta$

$y(x) = e^{(\alpha+i\beta)x}$ je kompleksno rešenje

$y(x) = e^{\alpha x}e^{i\beta x} = e^{\alpha x}(\cos\beta x + i\sin\beta x)$

Teorema: Ako je $y = u(x) + iv(x)$ rešenje (LHC), onda su i $y = u(x)$ i $y = v(x)$ rešenja (LHC).

Dokaz: **domaći**

Posledica: $y_1(x) = e^{\alpha x}\cos\beta x,\ y_2(x) = e^{\alpha x}\sin\beta x$ su rešenja (LHC)

Ona su lin. nezavisna:

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \beta e^{2\alpha x} \neq 0,\ (\beta \neq 0) \qquad \textbf{(proveriti)}$$

$$y_h = C_1 e^{\alpha x}\cos\beta x + C_2 e^{\alpha x}\sin\beta x$$

Primer: $y'' + y' - 2y = 0$

$$\lambda^2 + \lambda - 2 = 0 \;\Rightarrow\; \lambda_1 = 1,\; \lambda_2 = -2$$

$$y_1(x) = e^x,\; y_2(x) = e^{-2x} \;\Rightarrow\; y_h = C_1 e^x + C_2 e^{-2x}$$

e) Struktura opšteg rešenja (LNH)

$$\begin{aligned} (LNH) \qquad y'' + p(x)y' + q(x)y &= f(x) \\ (LH) \qquad y'' + p(x)y' + q(x)y &= 0 \end{aligned}$$

Teorema: Neka je $y_h = C_1 y_1(x) + C_2 y_2(x)$ opšte rešenje (LH), a $y_p(x)$ jedno partikularno rešenje (LNH). Tada je

$$y_{nh} = \underbrace{C_1 y_1(x) + C_2 y_2(x)}_{y_h} + y_p(x)$$

opšte rešenje (LNH).

Dokaz:

$$\begin{aligned} y_{nh} &= y_h + y_p \;\Rightarrow\; y'_{nh} = y'_h + y'_p, \\ y''_{nh} &= y''_h + y''_p \;\rightarrow (LNH) \end{aligned}$$

$$
\begin{aligned}
& y_h'' + y_p'' + p(x)(y_h' + y_p') + q(x)(y_h + y_p) = \\
= \; & \underbrace{y_h'' + py_h' + qy_h}_{0} + \underbrace{y_p'' + py_p' + qy_p}_{f(x)} = f(x)
\end{aligned}
$$

Napomena: Može se pokazati da je y_{nh} "**jako**" rešenje (sadrži sva rešenja)

Ostaje da se reši problem nalaženja y_p.

f) Metoda varijacije konstanti za (LNH)

$$
\begin{aligned}
(LNH) \qquad y'' + p(x)y' + q(x)y &= f(x) \\
(LH) \qquad y'' + p(x)y' + q(x)y &= 0
\end{aligned}
$$

$$
\begin{aligned}
y_h &= C_1 y_1(x) + C_2 y_2(x) \\
y_{nh} &= C_1(x) y_1(x) + C_2(x) y_2(x) \\
y_{nh}' &= C_1' y_1 + C_1 y_1' + C_2' y_2 + C_2 y_2'
\end{aligned}
$$

Neka je $\quad C_1' y_1 + C_2' y_2 = 0 \qquad (1)$

$$\begin{aligned} \Rightarrow y_{nh}' &= C_1 y_1' + C_2 y_2' \\ y_{nh}' &= C_1' y_1' + C_1 y_1'' + C_2' y_2' + C_2 y_2'' \to (LNH) \end{aligned}$$

$$\underbrace{C_1' y_1' + C_1 y_1'' + C_2' y_2' + C_2 y_2''}_{y_{nh''}} + p(x)(\underbrace{C_1 y_1' + C_2 y_2'}_{y_{nh}'}) +$$

$$+q(x)(\underbrace{C_1 y_1 + C_2 y_2}_{y_{nh}}) = f(x)$$

Medjutim,

$$C_1(y_1'' + p(x) y_1' + q(x) y_1) = 0, \; C_2(y_2'' + p(x) y_2' + q(x) y_2) = 0$$

pa je

$$C_1' y_1' + C_2' y_2' = f(x) \qquad (2)$$

Nepoznate funkcije $C_1(x)$ i $C_2(x)$ odredjuju se iz sistema (1), (2)

$$\begin{aligned} C_1' y_1 + C_2' y_2 &= 0 \\ C_1' y_1' + C_2' y_2' &= f(x) \end{aligned}$$

za koji je $det = W(x) \neq 0$.

$$C_1' = u_1(x),\ C_2' = u_2(x) \ \Rightarrow\ \begin{aligned} C_1(x) &= \int u_1(x)dx + C_1 \\ C_2(x) &= \int u_2(x)dx + C_2 \end{aligned}$$

$$\begin{aligned} y_{nh} &= (\int u_1(x)dx + C_1)y_1(x) + (\int u_2(x)dx + C_2)y_2(x) = \\ &= \underbrace{C_1y_1(x) + C_2y_2(x)}_{y_h} + \\ &+ \underbrace{(\int u_1(x)dx)y_1(x) + (\int u_2(x)dx)y_2(x)}_{y_p} \end{aligned}$$

Primer: $y'' - 4y' - 12y = e^{6x}$

$$y'' - 4y' - 12y = 0$$

$$\lambda^2 - 4\lambda - 12 = 0 \ \Rightarrow\ \lambda_1 = 6,\ \lambda_2 = -2$$

$$y_h = C_1e^{6x} + C_2e^{-2x}, \quad y_{nh} = C_1(x)e^{6x} + C_2(x)e^{-2x}$$

$$W(x) = \begin{vmatrix} e^{6x} & e^{-2x} \\ 6e^{6x} & -2e^{-2x} \end{vmatrix} = -8e^{4x}$$

$$
\begin{aligned}
C_1' e^{6x} + C_2' e^{-2x} &= 0 \\
6C_1' e^{6x} - 2C_2' e^{-2x} &= e^{6x}
\end{aligned}
$$

$$D_1 = \begin{vmatrix} 0 & e^{-2x} \\ e^{6x} & -2e^{-2x} \end{vmatrix} = -e^{4x} \Rightarrow C_1'(x) = \tfrac{1}{8}$$

$$D_2 = \begin{vmatrix} e^{6x} & 0 \\ 6e^{6x} & e^{6x} \end{vmatrix} = e^{12x} \Rightarrow C_2'(x) = -\tfrac{1}{8}e^{8x}$$

$$C_1(x) = \tfrac{1}{8}x + C_1, \quad C_2(x) = -\tfrac{1}{64}e^{8x} + C_2 \Rightarrow$$

$$y_{nh} = \underbrace{C_1 e^{6x} + C_2 e^{-2x}}_{y_h} + \underbrace{\frac{1}{8}xe^{6x} - \frac{1}{64}e^{6x}}_{y_p}$$

4. Linearna diferencijalna jednačina $n-$tog reda

a) Definicija. Egzistencija rešenja

$$(LNH) \quad y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = f(x)$$

$$(LH) \quad y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0$$

(KP) za (LNH) ili (LH) :

$$y(x_0) = y_0, \ldots, y^{(n-1)}(x_0) = y_0^{(n-1)}$$

<u>**Teorema**</u> **(Pikarova**, globalna):
Ako su $p_1(x), p_2(x), \ldots, p_n(x), f(x)$ neprekidne na (a, b), onda za bilo koje $x_0 \in (a, b)$ postoji rešenje (KP) na (a, b) i jedinstveno je.

b) Osobine rešenja (LH) jednačine $n-$tog reda

$y \equiv 0$ trivijalno rešenje (LH)

Teorema: Ako su $y_1(x), y_2(x), \ldots, y_n(x)$ rešenja (LH) jednačine, a $C_1, C_2, \ldots, C_n$ proizvoljne konstante,onda je i

$$y = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x)$$

rešenje te jednačine.

<u>**Definicija:**</u> Rešenja $y_1(x), y_2(x), \ldots, y_n(x)$ (LH) jednačine su **linearno zavisna** na (a, b) ako postoje

konstante $C_1, C_2, \ldots, C_n$, koje nisu sve jednake nuli, takve da je

$$C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x) = 0,\ \forall x \in (a, b)$$

Rešenja $y_1(x), y_2(x), \ldots, y_n(x)$ su **linearno nezavisna** na (a, b) ako važi implikacija

$$\begin{aligned} C_1 y_1(x) + \cdots + C_n y_n(x) &\equiv 0, x \in (a, b) \Rightarrow \\ & C_1 = \cdots = C_n = 0 \end{aligned}$$

Definicija: **Vronskijan** rešenja $y_1(x),\ y_2(x), \ldots, y_n(x)$ (LH) jednačine je determinanta

$$W(x) = \begin{vmatrix} y_1(x) & \ldots & y_n(x) \\ y_1'(x) & \ldots & y_n'(x) \\ \vdots & \ldots & \vdots \\ y_1^{(n-1)}(x) & \ldots & y_n^{(n-1)}(x) \end{vmatrix}$$

Teorema: Rešenja $y_1(x),\ y_2(x), \ldots, y_n(x)$ (LH) su linearno nezavisna na (a, b) **ako i samo ako** je $W(x) \neq 0,\ \forall x \in (a, b)$.

Teorema: Ako su $y_1(x)$, $y_2(x), \ldots, y_n(x)$ linearno nezavisna rešenja (LH) jednačine, tada njeno opšte rešenje

$$y_h = C_1 y_1(x) + \cdots + C_n y_n(x)$$

sadrži sva rešenja (LH) jednačine. ("**jako**" rešenje)

c) Linearna homogena jednačina $n-$tog reda sa konstantnim koeficijentima

$$(LHC) \qquad \boxed{y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y = 0}$$

$$p_i = \text{const};\ (a,b) = (-\infty, \infty)$$

Rešenje tražimo u obliku $y = e^{\lambda x} \Rightarrow$
$$\begin{aligned} y' &= \lambda e^{\lambda x} \\ &\vdots \\ y^{(n)} &= \lambda^n e^{\lambda x} \end{aligned}$$

$$\rightarrow (LHC):\ \lambda^n e^{\lambda x} + p_1 \lambda^{n-1} e^{\lambda x} + \cdots + p_n e^{\lambda x} = 0 \Rightarrow$$

$$(KJ) \quad \boxed{\lambda^n + p_1 \lambda^{n-1} + \cdots + p_n = 0} \quad \textbf{karakt. j-na}$$

$$\lambda^n + p_1 \lambda^{n-1} + \cdots + p_n = (\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$$

Teorema: Svakom $m-$tostrukom realnom korenu $\lambda = a$ odgovara m linearno nezavisnih rešenja

$$y = e^{ax}, y = xe^{ax}, \dots, y = x^{m-1}e^{ax}$$

Svakom $r-$tostukom konjugovano kompleksnom paru $\lambda = \alpha \pm i\beta$ odgovara $2r$ rešenja

$$\begin{aligned} y &= e^{\alpha x}\cos\beta x,\ y = xe^{\alpha x}\cos\beta x, \dots, y = x^{r-1}e^{\alpha x}\cos\beta x \\ y &= e^{\alpha x}\sin\beta x,\ \ y = xe^{\alpha x}\sin\beta x, \dots, y = x^{r-1}e^{\alpha x}\sin\beta x \end{aligned}$$

Svih n rešenja su linearno nezavisni.

Primer: $\qquad y^{IV} - 16y = 0$

$$\lambda^4 - 16 = 0$$
$$(\lambda^2 - 4)(\lambda^2 + 4) = 0$$
$$(\lambda - 2)(\lambda + 2)(\lambda - 2i)(\lambda + 2i) = 0$$

$$\begin{aligned} \lambda_1 = 2 &: \ y = e^{2x}, \qquad \lambda_2 = -2 : y = e^{-2x} \\ \lambda_3 = 2i &: \ (\alpha = 0, \beta = 2) \Rightarrow y = e^{0\cdot x}\cos 2x = \cos 2x \\ \lambda_4 = -2i &: \qquad\qquad\qquad\qquad y = \sin 2x \end{aligned}$$

$$y_h = C_1e^{2x} + C_2e^{-2x} + C_3\cos 2x + C_4\sin 2x$$

Domaći: $y^{VII} + 3y^{VI} + 3y^{V} + y^{IV} = 0$

d) Struktura opšteg rešenja (LNH)

$$\begin{aligned} (LNH) \quad y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y &= f(x) \\ (LH) \quad y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y &= 0 \end{aligned}$$

Teorema: Neka je $y_h = C_1 y_1(x) + \cdots + C_n y_n(x)$ **opšte rešenje** (LH)**, a** $y_p(x)$ **jedno partikularno rešenje** (LNH) **jednačine. Tada je**

$$y_{nh} = C_1 y_1(x) + \cdots + C_n y_n(x) + y_p(x)$$

opšte rešenje (LNH) **jednačine ('jako je** - sadrži sva rešenja te jednačine).

e) Metoda varijacije konstanti za (LNH) jednačinu

$$\begin{aligned} y_h &= C_1\, y_1(x) + \cdots + C_n\, y_n(x) \\ y_{nh} &= C_1(x) y_1(x) + \cdots + C_n(x) y_n(x) \end{aligned}$$

Funkcije $C_1(x), \ldots, C_n(x)$ se dobijaju iz sistema

$$\begin{aligned} C_1' y_1 &+ \cdots + C_n' y_n = 0 \\ C_1' y_1' &+ \cdots + C_n' y_n' = 0 \\ &\vdots \\ C_1' y_1^{(n-1)} &+ \cdots + C_n' y_n^{(n-1)} = f(x) \end{aligned}$$

$$\begin{aligned} det = W(x) \neq 0 \ \Rightarrow C_1' &= u_1(x), \ldots, C_n' = u_n(x) \ \Rightarrow \\ C_1(x) &= \int u_1(x)dx + C_1 \\ &\vdots \\ C_n(x) &= \int u_1(x)dx + C_n \end{aligned}$$

$$\begin{aligned} y_{nh} &= (\int u_1 dx + C_1)y_1 + \cdots + (\int u_n dx + C_n)y_n = \\ &= \underbrace{C_1 y_1 + \cdots + C_n y_n}_{y_h} + \underbrace{y_1 \int u_1 dx + \cdots + y_n \int u_n dx}_{y_p} \end{aligned}$$

f) Metoda neodredjenih koeficijenata za $(LNHC)$

$$(LNHC) \quad \boxed{y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y = f(x)}$$

$p_i = const, \; i = 1, \ldots, n$

$1^\circ \quad f(x) = e^{\alpha x}[P(x)\cos\beta x + Q(x)\sin\beta x]$

$P(x)$, $Q(x)$ polinomi, $st(P) = p$, $st(Q) = q$

$$\boxed{y_p(x) = x^s e^{\alpha x}\left[R(x)\cos\beta x + S(x)\sin\beta x\right]}$$

$R(x)$, $S(x)$ polinomi sa neodr. koeficijentima:

- $st(R) = st(S) = \max\{p, q\}$
- $s = \begin{cases} 0, \text{ ako } \alpha \pm i\beta \text{ nije koren } (KJ) \\ \text{red (višestrukost) korena } \alpha \pm i\beta \end{cases}$

Koeficijenti se odredjuju uvrštavanjem u samu jednačinu

Specijalni slučajevi:

• $\alpha = 0,\ \beta = 0 \ \Rightarrow\ f(x) = P(x)$

• $\alpha = 0, \beta \neq 0 \ \Rightarrow\ f(x) = P(x)\cos\beta x + Q(x)\sin\beta x$
($P \equiv 0$ ili $Q \equiv 0$ moguće)

• $\alpha \neq 0,\ \beta = 0 \ \Rightarrow\ f(x) = e^{\alpha x}P(x)$

Primeri:

• $f(x) = \sin 3x: \ \ \alpha = 0, \beta = 3;\ P \equiv 0,\ Q(x) = x,$
$st(P) = 0,\ st(Q) = 1$

$$y_p = x^s[(Ax + B)\cos 3x + (Cx + D)\sin 3x]$$

• $f(x) = x^2:\ \alpha = 0, \beta = 0, P(x) = x^2,\ st(P) = 2$

$$y_p = x^s(Ax^2 + Bx + C)$$

• $f(x) = e^x\cos x: \ \ \alpha = 1, \beta = 1;\ P \equiv 1, Q \equiv 0$
$st(P) = 0,\ st(Q) = 0$

$$y_p = x^s e^x(A\cos x + B\sin x)$$

$\mathbf{2^\circ} \quad f(x) = f_1(x) + \cdots + f_m(x)$

$$f_i(x) = e^{\alpha_i x}(P_i(x)\cos\beta_i x + Q_i(x)\sin\beta_i x),\ i = 1, \ldots, m$$

$y_p = y_{p_1} + \cdots + y_{p_m}$; $\quad y_{p_i}$ se dobija prema 1°

Domaći:

$$y^{V} - y^{IV} + y''' - y'' = e^x + \sin 2x$$

SISTEMI DIFERENCIJALNIH JEDNAČINA

1. Definicija. Egzistencija rešenja

$x_1, \ldots, x_n$ - nepoznate funkcije
t - nezavisno promenljiva (vreme)

Opšti oblik sistema diferencijalnih jednačina prvog reda:

$$\begin{aligned} F_1(t, x_1, x_1', \ldots, x_n, x_n') &= 0 \\ &\vdots \\ F_n(t, x_1, x_1', \ldots, x_n, x_n') &= 0 \end{aligned} \tag{1}$$

Normalni oblik:

$$\begin{aligned} x_1' &= f_1(t, x_1, \ldots, x_n) \\ &\vdots \\ x_n' &= f_n(t, x_1, \ldots, x_n), \end{aligned} \tag{2}$$

$x_1' = \frac{dx_1}{dt}, \ldots, x_n' = \frac{dx_n}{dt}$

Simetrični oblik:

$$\frac{dx_1}{f_1} = \cdots = \frac{dx_n}{f_n} = \frac{dt}{1} \tag{3}$$

Definicija: Skup funkcija $x_1(t), \ldots, x_n(t)$ je **rešenje** sistema (2) na (a, b) ako je

$$\begin{aligned} x_1'(t) &\equiv f_1(t, x_1(t), \ldots, x_n(t)) \\ &\vdots \\ x_n'(t) &\equiv f_n(t, x_1(t), \ldots, x_n(t)), \quad t \in (a, b) \end{aligned}$$

Definicija: Skup funkcija

$$\begin{aligned} x_1 &= x_1(t, C_1, \ldots, C_n) \\ &\vdots \\ x_n &= x_n(t, C_1, \ldots, C_n) \end{aligned}$$

je **opšte rešenje** sistema (2) na (a, b) ako identički zadovoljava (2) po t i $C_1, \ldots, C_n$ na (a, b)

Primer: Da li je familija

$$\begin{array}{lcl} x = C_1 + C_2e^{-t} & & x' = y - z \\ y = C_1 + C_3e^{t} & \text{o. r. sistema} & y' = z - x \\ z = C_1 + C_2e^{-t} + C_3e^{t} & & z' = y - x\,\text{?} \end{array}$$

$$x' \;=\; -C_2e^{-t} = (\underbrace{C_1 + C_3e^{t}}_{y}) - (\underbrace{C_1 + C_2e^{-t} + C_3e^{t}}_{z})$$

$$
\begin{aligned}
y' &= C_3e^t = (\underbrace{C_1 + C_2e^{-t} + C_3e^t}_{z}) - (\underbrace{C_1 + C_2e^{-t}}_{x}) \\
z' &= -C_2e^{-t} + C_3e^t = (\underbrace{C_1 + C_3e^t}_{y}) - (\underbrace{C_1 + C_2e^{-t}}_{x})
\end{aligned}
$$

Košijev problem: Neka $(t_0, x_1^0, \dots, x_n^0) \in R^{n+1}$. Postoji li rešenje sistema (2) koje zadovoljava početne uslove

$$x_1(t_0) = x_1^0, \dots, x_n(t_0) = x_n^0 \text{ ?}$$

Ako postoji, da li je jedinstveno?

Teorema (Pikarova): Neka je

$$P_{a,b} = \{(t, x_1, \dots, x_n) : |t - t_0| \le a, \quad |x_i - x_i^0| \le b, \quad i = 1, \dots, n\}$$

i neka su

1° $f_1, \dots, f_n$ neprekidne na $P_{a,b}$

2° $|f_i(t, x_1, \dots, x_n)| \le M, \ i = 1, \dots, n$

3° $|f_i(t, x_1, \dots, x_n) - f_i(t, \overline{x}_1, \dots, \overline{x}_n)| \le$
$\le L(|x_1 - \overline{x}_1| + \dots + |x_n - \overline{x}_n|), \ i = 1, \dots, n$

(Lipšicov uslov)

Tada na $(t_0 - h, t_0 + h)$, $h = \min\{a, b/M\}$ postoji rešenje $x_1 = x_1(t), \ldots, x_n = x_n(t)$ (KP) i ono je jedinstveno.

Napomena: Ako funkcije f_i, $i = 1, \ldots, n$ imaju ograničene parcijalne izvode na $P_{a,b}$, tj. $\left|\frac{\partial f_i}{\partial x_j}\right| \leq N$; $i, j = 1, \ldots, n$, onda one zadovoljavaju Lipšicov uslov.

2. Rešavanje (KP)

1° Približne metode

2° Iz opšteg rešenja:

$$\left.\begin{array}{l} x_1 = x_1(t, C_1, \ldots, C_n) \\ \vdots \\ x_n = x_n(t, C_1, \ldots, C_n) \end{array}\right\} \qquad \left.\begin{array}{l} x_1(t_0, C_1, \ldots, C_n) = x_1^0 \\ \vdots \\ x_n(t_0, C_1, \ldots, C_n) = x_n^0 \end{array}\right\}$$

$$\Rightarrow\ C_1^0, \ldots, C_n^0$$

Rešenje (KP): $$\left.\begin{array}{l} x_1 = x_1(t, C_1^0, \ldots, C_n^0) \\ \vdots \\ x_n = x_n(t, C_1^0, \ldots, C_n^0) \end{array}\right\}$$

Primer: U prethodnom primeru naći rešenje koje zadovoljava uslov $x(0) = 1$, $y(0) = 2$, $z(0) = 0$

$$\left.\begin{array}{l} x = C_1 + C_2 e^{-t} \\ y = C_1 + C_3 e^{t} \\ z = C_1 + C_2 e^{-t} + C_3 e^{t} \end{array}\right\} \Rightarrow \quad \left.\begin{array}{l} C_1 + C_2 = 1 \\ C_1 + C_3 = 2 \\ C_1 + C_2 + C_3 = 0 \end{array}\right\}$$

$\Rightarrow$ $C_1 = 3$, $C_2 = -2$, $C_3 = -1$. Traženo rešenje:

$$\left.\begin{array}{l} x = 3 - 2e^{-t} \\ y = 3 - e^{t} \\ z = 3 - 2e^{-t} - e^{t} \end{array}\right\}$$

<u>Mogućnosti</u> nalaženja opšteg rešenja:

a) Svodjenjem na diferencijalnu j-nu $n-$tog reda

b) Preko prvih integrala sistema

c) Metodama za rešavanje linearnih sistema

3. Veza sistema n diferencijalnih jednačina $n-$ tog reda sa jednom diferencijalnom jednačinom $n-$tog reda

a) Od (DJ) $n-$tog reda ka sistemu od n (DJ) :

$$x^{(n)} = f(t, \underset{\substack{\uparrow \\ x_1}}{x}, \underset{\substack{\uparrow \\ x_2}}{x'}, \ldots, \underset{\substack{\uparrow \\ x_n}}{x^{(n-1)}})$$

Smena: $x_1 = x, x_2 = x', \ldots, x_n = x^{(n-1)} \Rightarrow$

$$\begin{aligned} x_1' &= x' = x_2 \\ x_2' &= x'' = x_3 \\ &\vdots \\ x_n' &= x^{(n)} = f(t, x_1, \ldots, x_n) \end{aligned}$$

Dobijen je sistem:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ &\vdots \\ x_{n-1} &= x_n \\ x_n' &= f(t, x_1, \ldots, x_n) \end{aligned}$$

b) Od sistema n (DJ) ka (DJ) $n-$tog reda - Metoda uzastopnog diferenciranja

$$\begin{aligned} x_1' &= f_1(t, x_1, \ldots, x_n) \\ &\vdots \\ x_n' &= f_n(t, x_1, \ldots, x_n) \end{aligned} \qquad (S)$$

1° Jedna od jednačina, npr. prva, se diferencira $n-1$ put:

$$\begin{aligned} x_1' &= f_1(x_1, \ldots, x_n) \\ x_1'' &= \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial x_1} x_1' + \cdots + \frac{\partial f_1}{\partial x_n} x_n' = \\ &= \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial x_1} f_1 + \cdots + \frac{\partial f_1}{\partial x_n} f_n = \\ &= \varphi_2(t, x_1, \ldots, x_n) \\ &\vdots \\ x_1^{(n)} &= \varphi_n(t, x_1, \ldots, x_n) \end{aligned}$$

2° Prvih $n-1$ jednačina u 1° posmatramo kao sistem od $n-1$ jednačina sa $n-1$ nepoznatih $x_2, \ldots, x_n$:

$$\begin{aligned} x_1' &= f_1(t, x_1, \ldots, x_n) \\ x_1'' &= \varphi_2(t, x_1, \ldots, x_n) \\ &\vdots \\ x_1^{(n-1)} &= \varphi_{n-1}(t, x_1, \ldots, x_n) \end{aligned}$$

Prema **teoremi o implicitnoj funkciji**, ako je

$$\begin{vmatrix} \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial \varphi_2}{\partial x_2} & \cdots & \frac{\partial \varphi_2}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial \varphi_{n-1}}{\partial x_2} & \cdots & \frac{\partial \varphi_{n-1}}{\partial x_n} \end{vmatrix} \neq 0$$

sistem ima jedinstveno rešenje po $x_2, \ldots, x_n$

$\mathbf{3}^\circ$ Neka je rešenje dato sa

$$\begin{aligned} x_2 &= \lambda_2(t, x_1, x_1', \ldots, x_1^{(n-1)}) \\ &\vdots \\ x_n &= \lambda_n(t, x_1, x_1', \ldots, x_1^{(n-1)}) \end{aligned}$$

$\mathbf{4}^\circ$ Zamena $x_2, \ldots, x_n$ u poslednju j-nu u 1°:

$$\begin{aligned} x_1^{(n)} = \varphi_n(t, x_1, \; & \lambda_2 \; (t, x_1, \ldots, x_1^{(n-1)}), \ldots, \\ \ldots, \; & \lambda_n \; (t, x_1, \ldots, x_1^{(n-1)})) \end{aligned}$$

Dobijena jednačina je oblika

$$\boxed{x_1^{(n)} = \varphi(t, x_1, \ldots, x_1^{(n-1)})}$$

i predstavlja jednačinu $n-$tog reda po x_1

Važi: $x_1 = x_1(t)$ je rešenje j-ne 4° $\Rightarrow$

$$\begin{aligned} x_1 &= x_1(t) \\ x_2 &= \lambda_2(t, x_1(t), \ldots, x_1^{(n-1)}(t)) \\ &\vdots \\ x_n &= \lambda_n(t, x_1(t), \ldots, x_1^{(n-1)}(t)) \end{aligned}$$

je rešenje sistema (S)

Problem: rešivost sistema 2°, tj. efektivno nalaženje funkcija $\lambda_2, \ldots, \lambda_n$

Primer:

$$\begin{aligned} x' &= x - 2y \\ y' &= 4y + x \end{aligned}$$

1° $\quad x' = x - 2y$

$\quad\; x'' = x' - 2y' = x - 2y - 2(4y + x) = -x - 10y$

2° $\quad x' = x - 2y \;\Rightarrow \qquad (\dfrac{\partial f_1}{\partial y} = -2 \neq 0)$

3° $\quad y = \dfrac{x - x'}{2} \qquad (= \lambda_2(x, x'))$

4° $\quad x'' = -x - 10 \cdot \dfrac{x - x'}{2} = -6x + 5x'$

$$x'' - 5x' + 6x = 0$$

$$\lambda^2 - 5\lambda + 6 = 0 \;\Rightarrow\; \lambda_1 = 3,\, \lambda_2 = 2$$

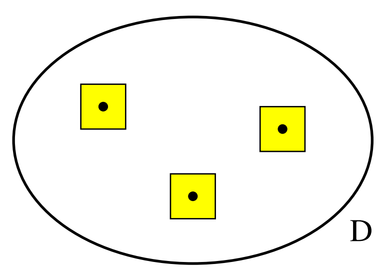
$$\left.\begin{array}{l} x = C_1 e^{3t} + C_2 e^{2t} \\ y = \frac{x - x'}{2} = -C_1 e^{3t} - \frac{C_2}{2} e^{2t} \end{array}\right\} \text{opšte rešenje sistema}$$

Primer: $\begin{array}{l} x' = y - z \\ y' = z - x \\ z' = y - x \end{array}$ **domaći**

Napomena: U opštem slučaju važi: sistem od n jednačina se može svesti na r jednačina reda $k_1, \ldots, k_r$, $k_1 + \cdots + k_r = n$

3. Prvi integrali sistema diferencijalnih jednačina

$$(S) \qquad \begin{array}{l} x_1' = f_1(t, x_1, \ldots, x_n) \\ \vdots \\ x_n' = f_n(t, x_1, \ldots, x_n) \end{array}$$



Ispunjeni su uslovi Pikarove teoreme u svakoj tački oblasti D

Definicija: Funkcija $\varphi(t, x_1, \ldots, x_n)$, neprekidno diferencijabilna i različita od konstante na oblasti D, je **integral** sistema (S) ako je

$$\varphi(t, x_1(t), \ldots, x_n(t)) \equiv const,\ t \in I$$

gde je $x_1(t), \ldots, x_n(t),\ t \in I$ proizvoljno rešenje sistema (S)

Primer: Ispitati da li je $\varphi = (x + y)e^{-2t}$ integral sistema

$$\begin{aligned} x' &= x - 2y \\ y' &= 4y + x \end{aligned}$$

"Proizvoljno rešenje" je opisano opštim rešenjem (prethodni rešeni primer):

$$\begin{aligned} x &= C_1 e^{3t} + C_2 e^{2t} \\ y &= -C_1 e^{3t} - \frac{C_2}{2} e^{2t} \end{aligned}$$

$$\begin{aligned} \varphi(x(t), y(t)) &= (C_1 e^{3t} + C_2 e^{2t} - C_1 e^{3t} - \frac{C_2}{2} e^{2t}) e^{-2t} = \\ &= \frac{C_2}{2} \equiv const \quad \surd \end{aligned}$$

Teorema: Funkcija $\varphi(t, x_1, \ldots, x_n)$, neprekidno diferencijabilna i različita od konstante na oblasti D, je **integral** sistema (S) **ako i samo ako** je

$$(*) \quad \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x_1} f_1 + \cdots + \frac{\partial \varphi}{\partial x_n} f_n = 0, \ (t, x_1, \ldots, x_n) \in D$$

Dokaz: $(\Rightarrow)$ Neka je $\varphi(t, x_1, \ldots, x_n)$ integral sistema (S), a $(t_0, x_1^0, \ldots, x_n^0)$ proizvoljna tačka oblasti D. Tada postoji rešenje

$$x_1 = x_1(t), \ldots, x_n = x_n(t), \ t \in I \quad (I = (t_0 - h, t_0 + h))$$

t. d.

$$x_1(t_0) = x_1^0, \ldots, x_n(t_0) = x_n^0. \qquad \text{(Pikar)}$$

Sledi:

$$\begin{aligned}
& \varphi(t, x_1(t), \ldots, x_n(t)) \equiv const, \ t \in I \\
\Leftrightarrow \ & \frac{d}{dt}\varphi(t, x_1(t), \ldots, x_n(t)) = 0, \quad t \in I \\
\Leftrightarrow \ & \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x_1} x_1'(t) + \cdots + \frac{\partial \varphi}{\partial x_n} x_n'(t) = 0, \ t \in I \\
\Leftrightarrow \ & \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x_1} f_1(t, x_1(t), \ldots, x_n(t)) + \cdots \\
\cdots \ & + \frac{\partial \varphi}{\partial x_n} f_n(t, x_1(t), \ldots, x_n(t)) = 0, \ t \in I
\end{aligned}$$

Specijalno, za $t = t_0$ je $(t_0, x_1(t_0), \ldots, x_n(t_0)) = (t_0, x_1^0, \ldots, x_n^0)$, pa $(*)$ važi u $(t_0, x_1^0, \ldots, x_n^0)$. Kako je to proizvoljna tačka, to $(*)$ važi na D.

$(\Leftarrow)$: Neka $(*)$ važi na D. Specijalno,tada $(*)$ važi u tačkama $(t, x_1(t), \ldots, x_n(t))$, $t \in I$, gde je $x_1(t), \ldots, x_n(t)$ proizvoljno rešenje sistema (S). Čitajući ekvivalencije unazad dobijamo $\varphi(t, x_1(t), \ldots, x_n(t)) \equiv const$, $t \in I$, tj. φ je integral sistema (S).

Primer: $\varphi = (x+y)e^{-2t}$, $\left.\begin{array}{l} x' = x - 2y \\ y' = 4y + x \end{array}\right\}$ $D = R^3$

$$\frac{\partial\varphi}{\partial t} + \frac{\partial\varphi}{\partial x} f_1 + \frac{\partial\varphi}{\partial y} f_2 = -2e^{-2t}(x+y) + e^{-2t}(x-2y) + e^{-2t}(4y+x) \equiv 0 \quad \surd$$

Definicija: Jednakost

$$(J) \qquad \varphi(t, x_1, \ldots, x_n) = C$$

gde je φ integral, a C konstanta, naziva se **prvi integral** sistema (S).

Primer: Proveriti da li su $x + y - z = C_1$, $x^2 + y^2 - z^2 = C_2$, $z^2 + xy - xz - yz = C_3$ prvi integrali sistema

$$\begin{aligned} x' &= y - z \\ y' &= z - x \qquad \textbf{(domaći)} \\ z' &= y - x \end{aligned}$$

Definicija: Prvi integrali $\varphi_1 = C_1, \dots, \varphi_k = C_k$ su <u>*nezavisni*</u> na D ako ni za jedno $i = 1, \dots, k$ ne postoji neprekidno diferencijabilna funkcija Φ i oblast $D' \subset D$ t.d. je

$$\varphi_i = \Phi(\varphi_1, \dots, \varphi_{i-1}, \varphi_{i+1}, \dots, \varphi_k)$$

na D'.

Teorema: Neka su $\varphi_1 = C_1, \dots, \varphi_n = C_n$ prvi integali sistema (S).

(i) Ako je

$$J = \begin{vmatrix} \frac{\partial \varphi_1}{\partial x_1} & \dots & \frac{\partial \varphi_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial \varphi_n}{\partial x_1} & \dots & \frac{\partial \varphi_n}{\partial x_n} \end{vmatrix} \neq 0 \text{ na } D \Rightarrow$$

$\varphi_1 = C_1, \dots, \varphi_n = C_n$ su nezavisni

(ii) Ako je $J \equiv 0$ na D $\Rightarrow$ $\varphi_1 = C_1, \ldots, \varphi_n = C_n$ su zavisni

Teorema: Ako je

$$rang \begin{bmatrix} \frac{\partial \varphi_1}{\partial x_1} & \cdots & \frac{\partial \varphi_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial \varphi_k}{\partial x_1} & \cdots & \frac{\partial \varphi_k}{\partial x_n} \end{bmatrix} = k \text{ na } D \Rightarrow$$

$\varphi_1 = C_1, \ldots, \varphi_k = C_k$ su nezavisni

4. Rešavanje sistema (DJ) pomoću prvih integrala

$$(S) \qquad \begin{aligned} x_1' &= f_1(t, x_1, \ldots, x_n) \\ &\vdots \\ x_n' &= f_n(t, x_1, \ldots, x_n) \end{aligned}$$

1° Ako je poznato n prvih integrala

$$\begin{aligned} \varphi_1(t, x_1, \ldots, x_n) &= C_1 \\ &\vdots \\ \varphi_n(t, x_1, \ldots, x_n) &= C_n \end{aligned}$$

koji su nezavisni, tj. $J \neq 0$ na oblasti D, sistem (S) je u **potpunosti rešen**

$a)$ Opšte rešenje:

$$\begin{aligned} x_1 &= x_1(t, C_1, \ldots, C_n) \\ &\vdots \\ x_n &= x_n(t, C_1, \ldots, C_n) \end{aligned}$$

$b)$ Rešenje (KP):

$$\begin{aligned} \varphi_1(t_0, x_1^0, \ldots, x_n^0) &= C_1^0 \\ &\vdots \\ \varphi_n(t_0, x_1^0, \ldots, x_n^0) &= C_n^0 \end{aligned}$$

Traženo rešenje (implicitno zadato):

$$\left.\begin{array}{c} \varphi_1(t, x_1, \ldots, x_n) = C_1^0 \\ \vdots \\ \varphi_n(t, x_1, \ldots, x_n) = C_n^0 \end{array}\right\} \Rightarrow \begin{array}{c} x = x_1(t) \\ \vdots \\ x_n = x_n(t) \end{array}$$

$\mathbf{2}^\circ$ Ako je poznato k prvih integrala

$$\begin{array}{c} \varphi_1(t, x_1, \ldots, x_n) = C_1 \\ \vdots \\ \varphi_k(t, x_1, \ldots, x_n) = C_k \end{array} \quad \text{t.d.} \quad \begin{vmatrix} \frac{\partial \varphi_1}{\partial x_1} & \cdots & \frac{\partial \varphi_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial \varphi_k}{\partial x_1} & \ldots & \frac{\partial \varphi_k}{\partial x_n} \end{vmatrix} \neq 0,$$

broj jednačina u sistemu se snižava za k:

$$\left.\begin{array}{c} x_1 = \lambda_1(t, x_{k+1}, \ldots, x_n, C_1, \ldots, C_k) \\ \vdots \\ x_k = \lambda_k(t, x_{k+1}, \ldots, x_n, C_1, \ldots, C_k) \end{array}\right\} \Rightarrow$$

$$\begin{aligned} x'_{k+1} &= f_{k+1}(t, \lambda_1(\cdot), \ldots, \lambda_k(\cdot), x_{k+1}, \ldots, x_n) \\ &\vdots \\ x'_n &= f_n(t, \lambda_1(\cdot), \ldots, \lambda_k(\cdot), x_{k+1}, \ldots, x_n) \end{aligned}$$

Dobija se $n-k$ jednačina sa $n-k$ nepoznatih funkcija

Primer:

$$\begin{aligned} x' &= y - z \\ y' &= z - x \\ z' &= y - x \end{aligned}$$

$$x + y - z = C_1 \ \Rightarrow\ x = C_1 - y + z \ \Rightarrow$$

$$\left.\begin{array}{l} y' = z - C_1 + y - z = -C_1 + y \\ z' = y - C_1 + y - y = 2y - C_1 - z \end{array}\right\} \text{ 2 j-ne}$$

5. Nalaženje prvih integrala

$$\frac{dx_1}{f_1(t, x_1, \ldots, x_n)} = \cdots = \frac{dx_n}{f_n(t, x_1, \ldots, x_n)} = \frac{dt}{1}$$

$t \leftrightarrow x_{n+1}$:

$$(*) \qquad \frac{dx_1}{X_1} = \cdots = \frac{dx_n}{X_n} = \frac{dx_{n+1}}{X_{n+1}}$$

gde je $X_i = X_i(x_1, \ldots, x_{n+1}), \quad i = 1, \ldots, n+1$

a) U $(*)$ učestvuju "jednostavni izrazi":

$$\frac{dx_i}{F_i(x_i, x_j)} = \frac{dx_j}{F_j(x_i, x_j)}$$

Obična DJ, opšte rešenje daje prvi integral

Primer: $\frac{dy}{y} = \frac{dy}{-x} = \frac{dt}{1}$

$$\frac{dx}{y} = \frac{dy}{-x} \Rightarrow x^2 + y^2 = C_1$$

$b)$ Osobina proporcije:

$$\frac{a_1}{b_1} = \cdots = \frac{a_n}{b_n} \Rightarrow \frac{k_1 a_1 + \cdots + k_n a_n}{k_1 b_1 + \cdots + k_n b_n} = \frac{a_i}{b_i},\ i = 1, \ldots, n$$

Nekad se može doći do veze oblika

$$\frac{du}{U(u,v)} = \frac{dv}{V(u,v)}, \quad \begin{array}{rcl} u &=& u(x_1, \ldots, x_{n+1}) \\ v &=& v(x_1, \ldots, x_{n+1}) \end{array}$$

Rešavanjem DJ dobija se prvi integral sistema

Primer: $\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{y-x} = \frac{dt}{1}$

$$\frac{dx-dz}{y-z-y+x} = \frac{dt}{1} \ \Rightarrow \ \frac{d(x-z)}{x-z} = dt \ \Rightarrow \ \frac{du}{u} = dt \ \Rightarrow$$

$$\Rightarrow \ u = C_1 e^t \ \Rightarrow \ (x-z)e^{-t} = C_1$$

Slično: $\frac{dy-dz}{z-y} = \frac{dt}{1} \ \Rightarrow \ (y-z)e^t = C_2$

$c)$ Osobina proporcije:

$$\frac{k_1a_1+\cdots+k_na_n}{0}=\frac{a_1}{b_1} \;\Rightarrow\; k_1a_1+\cdots+k_na_n=0$$

Nekad se može doći do veze oblika

$$\underbrace{P_1(x_1,\ldots,x_{n+1})dx_1+\cdots+P_{n+1}(x_1,\ldots,x_{n+1})dx_{n+1}}_{d\varphi(x_1,\ldots,x_{n+1})}=0$$

$$\Rightarrow\; \varphi(x_1,\ldots,x_{n+1})=C$$

Primer: $\frac{dx}{y-z}=\frac{dy}{z-x}=\frac{dz}{y-x}=\frac{dt}{1}\Rightarrow\frac{dx+dy-dz}{0}=$
$=\frac{dt}{1}\;\Rightarrow$

$$d(x+y-z)=0\;\Rightarrow x+y-z=C_1$$

6. Sistemi diferencijalnih jednačina višeg reda

$$\left\{\begin{array}{l} F_1(t,x_1,\ldots,x_1^{(k_1)},\ldots,x_n,\ldots,x_n^{(k_n)})=0\\ \vdots\\ F_n(t,x_1,\ldots,x_1^{(k_1)},\ldots,x_n,\ldots,x_n^{(k_n)})=0\end{array}\right. \quad\Rightarrow$$

$$\begin{cases} x_1^{(k_1)} = f_1(t, x_1, \dots, x_1^{(k_1-1)}, \dots, x_n, \dots, x_n^{(k_n-1)}) \\ \vdots \\ x_n^{(k_n)} = f_n(t, \underset{\underset{x_{11}}{\uparrow}}{x_1}, \dots, \underset{\underset{x_{k_1,1}}{\uparrow}}{x_1^{(k_1-1)}}, \dots, \underset{\underset{x_{1n}}{\uparrow}}{x_n}, \dots, \underset{\underset{x_{k_n,n}}{\uparrow}}{x_n^{(k_n-1)}}) \end{cases}$$

Smene:

$$\left. \begin{array}{l} x_{11} = x_1,\, x_{21} = x_1', \dots, x_{k_1,1} = x_1^{(k_1-1)} \\ \vdots \\ x_{1n} = x_n,\, x_{2n} = x_n', \dots, x_{k_n,n} = x_n^{(k_n-1)} \end{array} \right\} \Rightarrow$$

$$\begin{array}{ll} x_{11}' = x_{21} & x_{1n}' = x_{2n} \\ \vdots & \vdots \\ x_{k_1-1,1}' = x_{k_1,1} & x_{k_n-1,n}' = x_{k_n,n} \\ x_{k_1,1}' = f_1(t, x_{11}, \dots, x_{k_n,n}) & x_{k_n,n}' = f_n(t, x_{11}, \dots, x_{k_n,n}) \end{array}$$

Primer:

$$\begin{array}{rcl} x'' & = & 2x + 3y' \\ y''' & = & x' + y'' + e^t \end{array}$$

$$\begin{array}{l} x_1 = x \\ x_2 = x' \end{array} \quad \begin{array}{l} y_1 = y \\ y_2 = y' \\ y_3 = y'' \end{array} \Rightarrow \underbrace{\begin{array}{l} x_1' = x_2 \\ x_2' = 2x_1 + 3y_2 \end{array} \quad \begin{array}{l} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = x_2 + y_3 + e^t \end{array}}_{\text{5 j-na prvog reda}}$$

SISTEMI LINEARNIH DIFERENCIJALNIH JEDNAČINA

1. Definicija. Egzistencija rešenja

$$(L) \quad \begin{aligned} x_1' &= a_{11}(t)x_1 + \cdots + a_{1n}(t)x_n + b_1(t) \\ &\vdots \\ x_n' &= a_{n1}(t)x_1 + \cdots + a_{nn}(t)x_n + b_n(t) \end{aligned}$$

$$\begin{aligned} (LNH) &: \quad b_i(t) \neq 0 \text{ za bar jedno } i \\ (LH) &: \quad b_i(t) \equiv 0,\ i = 1, \ldots, n \\ (KP) &: \quad x_1(t_0) = x_1^0, \ldots, x_n(t_0) = x_n^0 \end{aligned}$$

Teorema (Pikar, globalna): Neka su $a_{ij}(t), b_i(t),\ i, j = 1, \ldots, n$ neprekidne na (c, d) i $t_0 \in (c, d)$. Tada za proizvoljne $x_1^0, \ldots, x_n^0$ postoji rešenje $x_1 = x_1(t), \ldots,$ $x_n = x_n(t)$ (KP) definisano na (c, d). To rešenje je jedinstveno.

Vektorski zapis sistema (L):

$$A = \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{bmatrix},\ X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},$$

$$X' = \frac{dX}{dt} = \begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix}, \; B(t) = \begin{bmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{bmatrix} :$$

(L) $$\boxed{\frac{dX}{dt} = A(t)X + B(t)}$$

(KP) $$X(t_0) = X_0, \; X_0 = \begin{bmatrix} x_1^0 \\ \vdots \\ x_n^0 \end{bmatrix}$$

Primer:

$$\begin{aligned} x_1' &= 5x_1 + 4x_2 + e^{2t} \\ x_2' &= 4x_1 + 5x_2 \end{aligned}$$

$$\underbrace{\begin{bmatrix} x_1' \\ x_2' \end{bmatrix}}_{\frac{dX}{dt}} = \underbrace{\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{X} + \underbrace{\begin{bmatrix} e^{2t} \\ 0 \end{bmatrix}}_{B}$$

2. Osobine rešenja linearnih homogenih sistema

(LH) $$\frac{dX}{dt} = A(t)X, \quad A(t) \text{ nepr. na } (c, d)$$

Teorema: Ako su

$$X_1 = \begin{bmatrix} x_{11}(t) \\ \vdots \\ x_{n1}(t) \end{bmatrix}, \ldots, X_n(t) = \begin{bmatrix} x_{1n}(t) \\ \vdots \\ x_{nn}(t) \end{bmatrix}$$

rešenja (LH) sistema, a $C_1, \ldots, C_n$ proizvoljne konstante, onda je

$$\begin{aligned} X(t) &= C_1 X_1(t) + \cdots + C_n X_n(t) = \\ &= \begin{bmatrix} C_1 x_{11}(t) + \cdots + C_n x_{1n}(t) \\ \vdots \\ C_1 x_{n1}(t) + \cdots + C_n x_{nn}(t) \end{bmatrix} \end{aligned} \tag{1}$$

rešenje (LH) sistema.

Dokaz:

$$\begin{aligned} \frac{dX}{dt} = C_1 \underbrace{\frac{dX_1}{dt}}_{AX_1} + \cdots + C_n \underbrace{\frac{dX_n}{dt}}_{AX_n} &= C_1 AX_1 + \cdots + C_n AX_n \\ &= A(C_1 X_1 + \cdots + C_n X_n) \\ &= AX \end{aligned}$$

Definicija: Rešenja $X_1(t), \ldots, X_n(t)$ su **linearno nezavisna** rešenja (LH) sistema na (c, d) ako važi implikacija

$$C_1X_1(t)+\cdots+C_nX_n(t) \equiv 0, t \in (c,d) \Rightarrow C_1 = \cdots = C_n = 0$$

Rešenja $X_1(t), \ldots, X_n(t)$ su **linearno zavisna** na (c, d) ako postoje konstante $C_1, \ldots, C_n$, koje nisu sve nule, takve da je

$$C_1X_1(t) + \cdots + C_nX_n(t) = 0,\ t \in (c,d)$$

Teorema: Neka su

$$X_1(t) = \begin{bmatrix} x_{11}(t) \\ \vdots \\ x_{n1}(t) \end{bmatrix}, \ldots, X_n(t) = \begin{bmatrix} x_{1n}(t) \\ \vdots \\ x_{nn}(t) \end{bmatrix}$$

rešenja (LH) sistema. Potreban i dovoljan uslov za linearnu nezavisnost tih rešenja na (c, d) dat je sa

$$W(t) = \begin{vmatrix} x_{11}(t) & \cdots & x_{1n}(t) \\ \vdots & & \vdots \\ x_{n1}(t) & \cdots & x_{nn}(t) \end{vmatrix} \neq 0,\ t \in (c,d)$$

Dokaz: ($\Rightarrow$) Neka su $X_1, \ldots, X_n$ linearno nezavisna rešenja (LH) sistema . Pretpostavimo, suprotno tvrdjenju teoreme, da

$$\exists t_0 \in (c,d) \text{ t.d. } W(t_0) = 0$$

Posmatrajmo sistem

$$\begin{aligned} &C_1 x_{11}(t_0) + \cdots + C_n x_{1n}(t_0) = 0 \\ &\vdots \\ &C_1 x_{n1}(t_0) + \cdots + C_n x_{nn}(t_0) = 0 \end{aligned}$$

Homoge sistem linearnih jednačina po $C_1, \ldots, C_n$ sa $det = W(t_0) = 0 \quad \Rightarrow \quad$ postoji netrivijalno rešenje $\overline{C}_1, \ldots, \overline{C}_n$. Neka je

$$X(t) = \overline{C}_1 X_1(t) + \cdots + \overline{C}_n X_n(t)$$

Važi:

- $X(t)$ je rešenje (LH) sistema
- $X(t_0) = O$
- Isti početni uslov zadovoljava i rešenje $X \equiv 0$

Iz Pikarove teoreme sledi

$$\overline{C}_1 X_1(t) + \cdots + \overline{C}_n X_n(t) \equiv 0,\ t \in (c,d)$$

Ovo je u kontradikciji sa linearnom nezavisnošću rešenja, a do koje je dovela pretpostavka $W(t_0) = 0$.

($\Leftarrow$): Neka je $W(t) \neq 0, \forall t \in (c,d)$. Pretpostavimo da je

$$C_1 X_1(t) + \cdots + C_n X_n(t) = 0,\ t \in (c,d)$$

Za fiksirano $t = t_0$ je:

$$C_1 X_1(t_0) + \cdots + C_n X_n(t_0) = 0,$$

t.j.

$$\begin{cases} C_1 x_{11}(t_0) + \cdots + C_n x_{1n}(t_0) = 0 \\ \quad \vdots \\ C_1 x_{n1}(t_0) + \cdots + C_n x_{nn}(t_0) = 0 \end{cases}$$

Homogeni sistem sa $det = W(t_0) \neq 0 \ \Rightarrow \ C_1 = 0, \ldots, C_n = 0$.

Napomena: Postoje dve mogućnosti: $W(t) \equiv 0$ ili $W(t) \neq 0, \forall t \in (c, d)$

Teorema: Neka su $X_1(t), \dots, X_n(t)$ linearno nezavisna rešenja (LH) sistema. Tada opšte rešenje $X_h = C_1X_1(t) + \cdots + C_nX_n(t)$ sadrži sva rešenja tog sistema.

Dokaz: Neka je $X(t)$ proizvoljno rešenje, a $t_0 \in (c, d)$ fiksirano. Tražimo $C_1, \dots, C_n$ t.d.

$$C_1X_1(t_0) + \cdots + C_nX_n(t_0) = X(t_0),$$

t.j.

$$\begin{cases} C_1x_{11}(t_0) + \cdots + C_nx_{1n}(t_0) = x_1(t_0) \\ \quad\vdots \\ C_1x_{n1}(t_0) + \cdots + C_nx_{nn}(t_0) = x_n(t_0) \end{cases}$$

Nehomogen sistem, $det = W(t_0) \neq 0 \;\Rightarrow\;$ postoji jedinstveno rešenje $C_1^0, \dots, C_n^0$.

Rešenja $X(t)$ i $C_1^0 X_1(t) + \cdots + C_n^0 X_n(t)$ zadovoljavaju isti početni uslov u $t_0 \Rightarrow$ (Pikar)

$$X(t) \equiv C_1^0 X_1(t) + \cdots + C_n^0 X_n(t),\ t \in (c, d)$$

Definicija: Matrica $\Phi(t) = [x_{ij}(t)]_{n \times n}$ je **fundamentalna matrica** sistema (LH) ako su kolone matrice $\Phi(t)$ linearno nezavisna rešenja (LH)

$$\Phi(t) = \begin{bmatrix} x_{11}(t) & \dots & x_{1n}(t) \\ \vdots & & \vdots \\ x_{1n}(t) & \dots & x_{nn}(t) \end{bmatrix}$$
$$\begin{matrix} \uparrow & \qquad & \uparrow \\ X_1(t) & & X_n(t) \end{matrix}$$

Napomena: Kolone matrice $\Phi(t)$ su rešenja (LH) sistema **ako i samo ako** $\Phi(t)$ zadovoljava pridruženu matričnu diferencijalnu jednačinu

$$(M) \qquad \frac{d\Phi}{dt} = A(t)\Phi$$

Primer: $\Phi(t) = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}, \quad \begin{matrix} x_1' = a_{11}x_1 + a_{12}x_2 \\ x_2' = a_{21}x_1 + a_{22}x_2 \end{matrix}$

$$(M) \quad \begin{bmatrix} x_{11}' & x_{12}' \\ x_{21}' & x_{22}' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \Leftrightarrow$$

$$\left.\begin{matrix} x_{11}' = a_{11}x_{11} + a_{12}x_{21} \\ x_{21}' = a_{21}x_{11} + a_{22}x_{21} \end{matrix}\right\} \text{ prva kolona zadov. sistem}$$

$$\left.\begin{matrix} x_{12}' = a_{11}x_{12} + a_{12}x_{22} \\ x_{22}' = a_{21}x_{12} + a_{22}x_{22} \end{matrix}\right\} \text{ druga kolona zadov. sistem}$$

Teorema: Matrica $\Phi(t)$ je fundamentalna matrica sistema (LH) **ako i samo ako** je $\Phi(t)$ rešenje (M) i ako je $det\Phi(t) \neq 0$, $\forall t \in (c, d)$.

Dokaz:

$$\begin{matrix} \Phi(t) \text{ je rešenje } (M) & \Leftrightarrow & \text{kolone su rešenja } (LH) \\ det\Phi(t) = W(t) \neq 0 & \Leftrightarrow & \text{rešenja su lin. nezavisna} \end{matrix}$$

Teorema: Ako je $\Phi(t)$ fundamentalna matrica, a P nesingularna matrica, onda je $\Psi(t) = \Phi(t)P$ fundamentalna matrica.

Dokaz:

$$\begin{aligned} \frac{d\Psi}{dt} &= \frac{d\Phi}{dt}P = A(t)\Phi P = A(t)\Psi \\ det\Psi &= det\Phi \cdot detP \neq 0 \end{aligned}$$

Napomena:

$$\begin{aligned} \Phi(t)C &= \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix} \begin{bmatrix} C_1 \\ \vdots \\ C_n \end{bmatrix} = \\ &= \begin{bmatrix} C_1 x_{11} + \cdots + C_n x_{1n} \\ \vdots \\ C_n x_{n1} + \cdots + C_n x_{nn} \end{bmatrix} = \\ &= C_1 \begin{bmatrix} x_{11} \\ \vdots \\ x_{n1} \end{bmatrix} + \cdots + C_n \begin{bmatrix} x_{1n} \\ \vdots \\ x_{nn} \end{bmatrix} = \\ &= C_1 X_1 + \cdots + C_n X_n \end{aligned}$$

Načini zapisa opšteg rešenja:

$X_h = \Phi(t)C$ — **matrični** zapis

$X_h = C_1 X_1 + \cdots + C_n X_n$ — **vektorski** zapis

$$\left.\begin{aligned} x_1 &= C_1 x_{11} + \cdots + C_n x_{1n} \\ &\vdots \\ x_n &= C_1 x_{n1} + \cdots + C_n x_{nn} \end{aligned}\right\}$$ **skalarni** zapis

Primer: Da li je matrica

$$\Phi(t) = \begin{bmatrix} -e^t & e^{9t} \\ e^t & e^{9t} \end{bmatrix}$$

fundamentalna matrica sistema

$$\begin{aligned} x_1' &= 5x_1 + 4x_2 \\ x_2' &= 4x_1 + 5x_2\,? \end{aligned}$$

$$A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix};\ X_1 = \begin{bmatrix} -e^t \\ e^t \end{bmatrix};\ X_2 = \begin{bmatrix} e^{9t} \\ e^{9t} \end{bmatrix}$$

$$\begin{aligned} \frac{d\Phi}{dt} &= \begin{bmatrix} -e^t & 9e^{9t} \\ e^t & 9e^{9t} \end{bmatrix} \\ A \cdot \Phi(t) &= \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} -e^t & e^{9t} \\ e^t & e^{9t} \end{bmatrix} = \begin{bmatrix} -e^t & 9e^{9t} \\ e^t & 9e^{9t} \end{bmatrix} \ \checkmark \end{aligned}$$

$$det\, \Phi(t) = -2e^{10t} \neq 0,\ t \in (-\infty, \infty)$$

$$\begin{aligned} X_h = \begin{bmatrix} -e^t & e^{9t} \\ e^t & e^{9t} \end{bmatrix} &= \begin{bmatrix} -e^t & 9e^{9t} \\ e^t & 9e^{9t} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \\ &= C_1 \begin{bmatrix} -e^t \\ e^t \end{bmatrix} + C_2 \begin{bmatrix} e^{9t} \\ e^{9t} \end{bmatrix} \end{aligned}$$

$$\left.\begin{array}{l} x_1 = -C_1 e^t + C_2 e^{9t} \\ x_2 = \;\; C_1 e^t + C_2 e^{9t} \end{array}\right\} \text{opšte rešenje}$$

Napomena: Rešenje (KP) $X(t_0) = X_0$, $t_0 \in (c, d)$:

$$X(t) = \Phi(t)\, C$$

$$\underbrace{X(t_0)}_{X_0} = \Phi(t_0)\, C \;\Rightarrow\; C = \Phi^{-1}(t_0) X_0$$

$$\boxed{X(t) = \Phi^{-1}(t_0) X_0}$$

3. Homogeni linearni sistemi DJ sa konstantnim koeficijentima

$$(LHC) \qquad \frac{dX}{dt} = AX,$$

$A = const$, $(c, d) = (-\infty, \infty)$, ili, skalarno:

$$\begin{aligned} x_1' &= a_{11}x_1 + \cdots + a_{1n}x_n \\ &\vdots \\ x_n' &= a_{n1}x_1 + \cdots + a_{nn}x_n \end{aligned}$$

Reš. tražimo u obliku: $x_1 = A_1 e^{\lambda t}, \ldots, x_n = A_n e^{\lambda t} \;\Rightarrow$

$$x_1' = A_1 \lambda e^{\lambda t}, \ldots, x_n' = A_n \lambda e^{\lambda t} \to (LHC)$$

$$\begin{aligned} \lambda A_1 e^{\lambda t} &= a_{11} A_1 e^{\lambda t} + \cdots + a_{1n} A_n e^{\lambda t} \\ &\vdots \\ \lambda A_n e^{\lambda t} &= a_{n1} A_1 e^{\lambda t} + \cdots + a_{nn} A_n e^{\lambda t} \end{aligned} \quad \Leftrightarrow$$

$$(S) \qquad \begin{aligned} (a_{11} - \lambda) A_1 + \cdots + a_{1n} A_n &= 0 \\ &\vdots \\ a_{n1} A_1 + \cdots + (a_{nn} - \lambda) A_n &= 0 \end{aligned}$$

Homogeni sistem lin. alg. jednačina po $A_1, \ldots, A_n$. Netrivijalno rešenje postoji ako i samo ako je $det(S) = 0$, tj.

$$(KJ) \qquad \begin{vmatrix} a_{11} - \lambda & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0,$$

odnosno

$$(KJ) \qquad det(A - \lambda I) = 0$$

Polinom $n-$tog stepena po λ čiji su koreni $\lambda_1, \ldots, \lambda_n$.

$1^\circ \quad \lambda = a$ je realan jednostruk koren:

$$x_1 = A_1 e^{at}, \ldots, x_n = A_n e^{at} \quad \text{rešenje}$$

$A_1, \ldots, A_n$ se odredjuju iz sistema (S) za $\lambda = a$.

Pokazuje se: rang matrice sistema (S) je $n-1$ (1 slobodna, $n-1$ vezanih promenljivih)

Primer: $\begin{array}{l} x_1' = 5x_1 + 4x_2 \\ x_2' = 4x_1 + 5x_2 \end{array} \qquad A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$

$$\begin{vmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 9 = 0 \Rightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 9 \end{cases}$$

$$\lambda_1 = 1: \quad \begin{array}{l} x_1 = A_1 e^t \\ x_2 = A_2 e^t \end{array} ; \qquad \left.\begin{array}{l} 4A_1 + 4A_2 = 0 \\ \boxed{4A_1 + 4A_2 = 0} \end{array}\right\} (S)$$

$$A_2 = 1 \text{ (proizv.)} \Rightarrow A_1 = -1, \quad X_1 = \begin{bmatrix} -e^t \\ e^t \end{bmatrix}$$

$$\lambda_2 = 9: \quad \begin{array}{l} x_1 = A_1 e^{9t} \\ x_2 = A_2 e^{9t} \end{array} ; \qquad \left.\begin{array}{l} -4A_1 + 4A_2 = 0 \\ \boxed{4A_1 - 4A_2 = 0} \end{array}\right\} (S)$$

$$A_2 = 1 \text{ (proizv.)} \Rightarrow A_1 = 1, \quad X_2 = \begin{bmatrix} e^{9t} \\ e^{9t} \end{bmatrix}$$

$$\Phi(t) = \begin{bmatrix} -e^{t} & e^{9t} \\ e^{t} & e^{9t} \end{bmatrix}; \quad X_h = \Phi(t)C$$

2° $\lambda = \alpha \pm i\beta$ jednostruk kompleksan par:

$$x_1 = A_1 e^{(\alpha+i\beta)t}, \ldots, x_n = A_n e^{(\alpha+i\beta)t} \quad \text{rešenje}$$

$A_1, \ldots, A_n$ se odredjuju iz sistema (S) koji, posle uvrštavanja λ, ima 1 slobodnu i $n-1$ vezanih promenljivih. Rešenja su kompleksna.
Razdvajanjem Re i Im delova dobijamo 2 realna rešenja

Primer: $\begin{aligned} x' &= -2y \\ y' &= \ \ 2x \end{aligned}$ **domaći**

3° $\lambda = a$ je realan koren reda k:

$$x_1 = P_1(t)e^{at}, \ldots, x_n = P_n(t)e^{at} \quad \text{rešenje}$$

$P_1(t), \ldots, P_n(t)$ polinomi stepena $k-1$ sa neodredjenim koeficijentima $\rightarrow$ (LHC) : sistem od $n \cdot k$ jedn. sa $n \cdot k$ nepoznatih

Pokazuje se: rang matrice sistema je $nk - k$ (k slobodnih, $nk - k$ vezanih promenljivih)

Izbor slobodnih promenljivih:

$$\left.\begin{array}{l} \underbrace{\begin{array}{cccc} 1 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{array}}_{k} \begin{array}{l} \to \text{ vezane } \to X_1 \\ \\ \to \text{ vezane } \to X_k \end{array} \end{array}\right\} \text{linearno nez. reš.}$$

Primer: $\begin{array}{l} x' = y \\ y' = -x + 2y \end{array} \qquad A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$

$$\begin{vmatrix} -\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 1 = 0 \Rightarrow \lambda_1 = \lambda_2 = 1$$

$$\begin{array}{l} x = (A_1 + A_2 t)e^t \\ y = (B_1 + B_2 t)e^t \end{array} \Rightarrow \left.\begin{array}{l} x' = A_2 e^t + (A_1 + A_2 t)e^t \\ y' = B_2 e^t + (B_1 + B_2 t)e^t \end{array}\right\} \overset{(LHC)}{\longrightarrow}$$

$$\begin{array}{rcl} (A_1 + A_2 + A_2 t)e^t & = & (B_1 + B_2 t)e^t \\ (B_1 + B_2 + B_2 t)e^t & = & -(A_1 + A_2 t)e^t + 2(B_1 + B_2 t)e^t \end{array}$$

$$\left.\begin{array}{rcl} A_1 + A_2 - B_1 & & = 0 \\ A_2 & - B_2 & = 0 \\ A_1 \quad - & B_1 + B_2 & = 0 \\ A_2 & - B_2 & = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{rcl} A_1 + A_2 - B_1 & & = 0 \\ A_2 & - B_2 & = 0 \end{array}\right\}$$

a) $B_1 = 1,\ B_2 = 0:\ A_2 = 0, A_1 = 1 \Rightarrow X_1 = \begin{bmatrix} e^t \\ e^t \end{bmatrix}$

b) $B_1 = 0,\ B_2 = 1:\ A_2 = 1, A_1 = -1 \Rightarrow$

$$X_2 = \begin{bmatrix} (-1+t)e^t \\ te^t \end{bmatrix}$$

$$\Phi(t) = \begin{bmatrix} e^t & (-1+t)e^t \\ e^t & te^t \end{bmatrix}, \qquad X_h = \Phi(t)C$$

4° $\lambda = \alpha \pm i\beta$ konjugovano kompl. par reda r:

$$x_1 = P_1(t)e^{(\alpha+i\beta)t}, \ldots, x_n = P_n(t)e^{(\alpha+i\beta)t}$$

Na isti način kao u 3° dobija se r linearno nezavisnih kompleksnih rešenja.

Razdvajanjem Re i Im delova dobija se $2r$ linearno nezavisnih rešenja.

4. Nehomogeni sistemi

$(LNH)\quad \frac{dX}{dt} = A(t)X + B(t); \quad A,\ B$ nepr. na (c, d)

$(LH)\quad \frac{dX}{dt} = A(t)X$

Teorema: Neka su $X_1(t), \dots, X_n(t)$ linearno nezavisna rešenja (LH), a $X_p(t)$ jedno rešenje (LNH). Tada opšte rešenje

$$X_{nh} = C_1 X_1(t) + \cdots + C_n X_n(t) + X_p(t)$$

sadrži sva rešenja (LNH).

Dokaz: a) X_{nh} je rešenje (LNH):

$$\begin{aligned}
\frac{dX_{nh}}{dt} &= C_1 \underbrace{\frac{dX_1}{dt}}_{AX_1} + \cdots + C_n \underbrace{\frac{dX_n}{dt}}_{AX_n} + \underbrace{\frac{dX_p}{dt}}_{AX_p + B} = \\
&= C_1 AX_1 + \cdots + C_n AX_n + AX_p + B = \\
&= A(C_1 X_1 + \cdots + C_n X_n + X_p) + B = \\
&= AX_{nh} + B
\end{aligned}$$

b) Ako je $X(t)$ proizvoljno rešenje (LNH), slično kao u homogenom slučaju, pokazuje se da postoje $C_1^\circ, \ldots, C_n^\circ$ t.d.

$$X(t) = C_1^\circ X_1(t) + \cdots + C_n^\circ X_n(t) + X_p(t)$$

5. Metoda varijacije konstanti

$$X_{nh} = C_1(t)X_1(t) + \cdots + C_n(t)X_n(t)$$

$$\begin{aligned}\frac{dX_{nh}}{dt} &= C_1'X_1 + C_1 \underbrace{X_1'}_{AX_1} + \cdots + C_n'X_n + C_n \underbrace{X_n'}_{AX_n} = \\ &= C_1'X_1 + \cdots + C_n'X_n + A(t)(C_1X_1 + \cdots + C_nX_n) \\ &= C_1'X_1 + \cdots + C_n'X_n + A(t)X_{nh} \to (LNH):\end{aligned}$$

$$C_1'X_1 + \cdots + C_n'X_n + A(t)X_{nh} = A(t)X_{nh} + B(t) \Rightarrow$$

$$C_1'X_1 + \cdots + C_n'X_n = B(t)$$

tj.

$$C_1' \begin{bmatrix} x_{11} \\ \vdots \\ x_{n1} \end{bmatrix} + \cdots + C_n' \begin{bmatrix} x_{1n} \\ \vdots \\ x_{nn} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix},$$

$$\begin{cases} x_{11}C_1' + \cdots + x_{1n}C_n' = b_1 \\ \vdots \\ x_{n1}C_1' + \cdots + x_{nn}C_n' = b_n \end{cases}$$

Sistem lin. alg. jednačina, $det = W(t) \neq 0 \Rightarrow$
$C_1' = v_1(t), \ldots, C_n' = v_n(t)$ jedinstv. reš. $\Rightarrow$

$$C_1(t) = \int v_1(t)\,dt + C_1, \ldots, C_n(t) = \int v_n(t)\,dt + C_n \Rightarrow$$

$$\begin{aligned} X_{nh} &= \left(\int v_1\,dt, +C_1\right)X_1 + \cdots + \left(\int v_n\,dt, +C_n\right)X_n = \\ &= \underbrace{C_1X_1 + \cdots + C_nX_n}_{X_h} + \\ &+ \underbrace{\left(\int v_1\,dt\right)X_1 + \cdots + \left(\int v_n\,dt\right)X_n}_{X_p} \end{aligned}$$

$$\begin{matrix} x_1' = 5x_1 + 4x_2 + e^{2t} \\ x_2' = 4x_1 + 5x_2 \end{matrix} \qquad \begin{matrix} x_1' = 5x_1 + 4x_2 \\ x_2' = 4x_1 + 5x_2 \end{matrix} \quad \text{(rešen ranije):}$$

$$\Phi(t) = [X_1(t)\ X_2(t)] = \begin{bmatrix} -e^t & e^{9t} \\ e^t & e^{9t} \end{bmatrix},\ W = det\Phi = -2e^{10t}$$

$$\begin{aligned} -e^t C_1' + e^{9t} C_2' &= e^{2t} \\ e^t C_1' + e^{9t} C_2' &= 0 \end{aligned}$$

$$D_1 = \begin{vmatrix} e^{2t} & e^{9t} \\ 0 & e^{9t} \end{vmatrix}, \quad D_2 = \begin{vmatrix} -e^t & e^{2t} \\ e^t & 0 \end{vmatrix}$$

$$C_1'(t) = \frac{D_1}{W} = -\frac{1}{2}e^t \quad \Rightarrow \quad C_1(t) = -\frac{1}{2}e^t + C_1$$

$$C_2'(t) = \frac{D_2}{W} = \frac{1}{2}e^{-7t} \quad \Rightarrow \quad C_2(t) = -\frac{1}{14}e^{-7t} + C_2$$

$$\begin{aligned} X_{nh} &= (-\frac{1}{2}e^t + C_1) \begin{bmatrix} -e^t \\ e^t \end{bmatrix} + (-\frac{1}{14}e^{-7t} + C_2) \begin{bmatrix} e^{9t} \\ e^{9t} \end{bmatrix} = \\ &= C_1 \underbrace{\begin{bmatrix} -e^t \\ e^t \end{bmatrix}}_{X_1} + C_2 \underbrace{\begin{bmatrix} e^{9t} \\ e^{9t} \end{bmatrix}}_{X_2} + \underbrace{\begin{bmatrix} \frac{3}{7}e^{2t} \\ -\frac{4}{7}e^{2t} \end{bmatrix}}_{X_p} \end{aligned}$$

6. Matrični zapis metode varijacije konstanti

$$X_h = \Phi(t)C, \qquad X_{nh} = \Phi(t)C(t)$$

Sledi:

$$\frac{dX_{nh}}{dt} = \underbrace{\frac{d\Phi}{dt}}_{A(t)\Phi(t)} C + \Phi\frac{dC}{dt} = \underbrace{A(t)\Phi(t)}C(t) + B(t) \Rightarrow$$

$$\Phi\frac{dC}{dt} = B(t)$$

$$\frac{dC}{dt} = \Phi^{-1}B(t)$$

$$C(t) = \int \Phi^{-1}(t)B(t)\,dt + C$$

$$\begin{aligned} X_{nh} &= \Phi(t)[\int \Phi^{-1}(t)B(t)\,dt + C] \\ &= \underbrace{\Phi(t)C}_{X_h} + \underbrace{\Phi(t) + \Phi(t)[\int \Phi^{-1}(t)B(t)\,dt]}_{X_p} \end{aligned}$$

7. Matrični zapis rešenja (KP)

$$\frac{dX}{dt} = A(t)X + B(t), \quad X(t_0) = X_0$$

$$X(t) = \Phi(t)C(t)$$

1) $X(t_0) = \Phi(t_0)C(t_0) = X_0 \;\Rightarrow\; C(t_0) = \Phi^{-1}(t_0)X_0$

2) $C(t)$ zadovoljava

$$\frac{dC}{dt} = \Phi^{-1}(t)B(t)$$

Integracijom od t_0 do t:

$$\int_{t_0}^{t} dC = \int_{t_0}^{t} \Phi^{-1}(t)B(t)\,dt \;\Rightarrow$$

$$C(t) - C(t_0) = \int_{t_0}^{t} \Phi^{-1}(t)B(t)\,dt$$

$$C(t) = \Phi^{-1}(t_0)X_0 + \int_{t_0}^{t} \Phi^{-1}(t)B(t)\,dt$$

$$X(t) = \Phi(t)[\Phi^{-1}(t_0)X_0 + \int_{t_0}^{t} \Phi^{-1}(t)B(t)\,dt]$$

8. Stabilnost sistema LDJ sa konstantnim koeficijentima

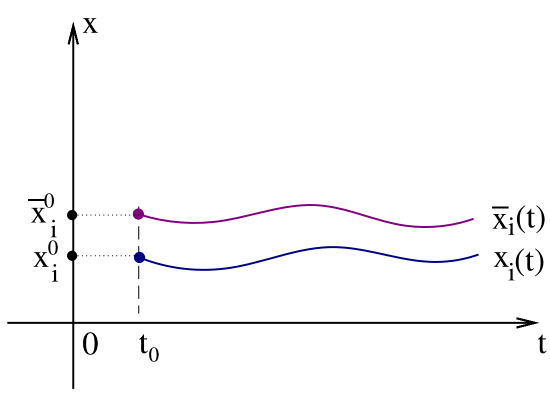
$$\left.\begin{array}{l} x_1' = a_{11}x_1 + \cdots + a_{1n}x_n + b_1(t) \\ \vdots \\ x_n' = a_{n1}x_1 + \cdots + a_{nn}x_n + b_n(t) \end{array}\right\}$$

(KP) $\qquad x_1(t_0) = x_1^0, \ldots, x_n(t_0) = x_n^0$

Reš. $\qquad x_1 = x_1(t), \ldots, x_n = x_n(t)$

(KP') $\qquad x_1(t_0) = \bar{x}_1^0, \ldots, x_n(t_0) = \bar{x}_n^0$

Reš. $\qquad x_1 = \bar{x}_1(t), \ldots, x_n = \bar{x}_n(t)$



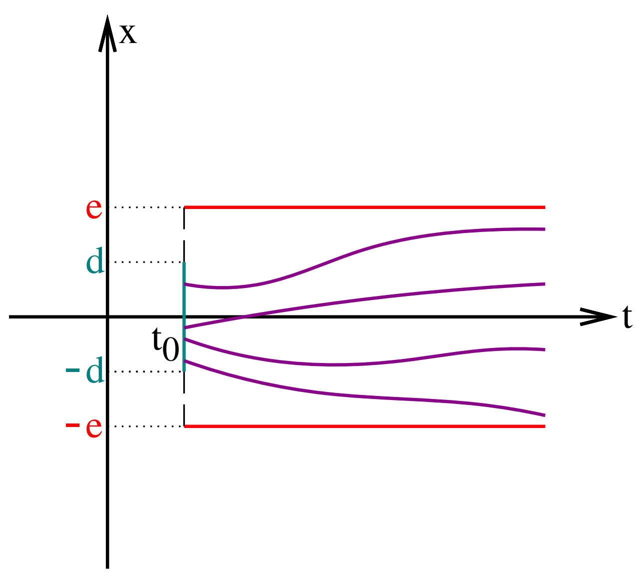
Pitanje: Ako su x_i^0 i $\bar{x}_i^0$ $i = 1, \ldots, n$ "bliski", da li će rešenja ostati "bliska" u budućnosti?

Ispitivanje stabilnosti proizvoljnog rešenja se svodi na ispitivanje stabilnosti trivijalnog rešenja homogenog sistema

$$\begin{aligned} x_1' &= a_{11}x_1 + \cdots + a_{1n}x_n \\ &\vdots \\ x_n' &= a_{n1}x_1 + \cdots + a_{nn}x_n \end{aligned}$$

Definicija: Trivijalno rešenje je **stabilno** ako

$$(\forall \varepsilon > 0)(\exists \delta > 0)(|\bar{x}_i^0| < \delta,\ i = 1, \ldots, n \Rightarrow |\bar{x}_i(t)| < \varepsilon,\ i = 1, \ldots, n;\ t \geq t_0)$$

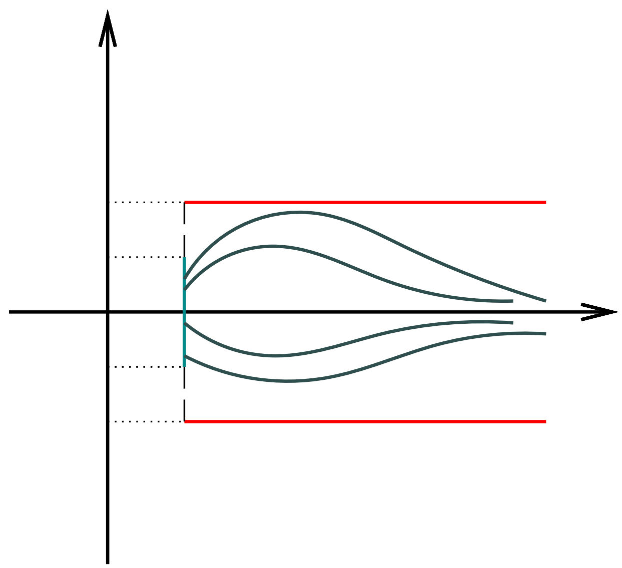


Definicija: Trivijalno rešenje je **asimptotski stabilno** ako je:

$a)$ stabilno

$b)$ $\exists \delta_0 > 0 : |\bar{x}_i^0| < \delta_0 \Rightarrow \lim_{t \to \infty} \bar{x}_i(t) = 0,$

$$i = 1, \ldots, n$$



Teorema 1. Trivijalno rešenje je asimptotski stabilno ako i samo ako svi koreni (KJ) imaju negativne realne delove.

Teorema 2. Ako bar jedan koren (KJ) ima pozitivan realni deo, onda trivijalno rešenje nije stabilno.

9. Matrični eksponent

$$e^a = \sum_{k=0}^{\infty} \frac{a^k}{k!} = 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \cdots \quad (Mc\ Laurin)$$

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots; \quad det\, e^A \neq 0, \forall A$$

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = I + At + \frac{A^2}{2!}t^2 + \frac{A^3}{3!}t^3 + \cdots$$

$$\begin{aligned} \frac{d}{dt}e^{At} &= \frac{d}{dt}(I + At + \frac{A^2}{2!}t^2 + \frac{A^3}{3!}t^3 + \cdots) = \\ &= A + \frac{A^2}{2!} \cdot 2t + \frac{A^3}{3!} \cdot 3t^2 + \cdots \\ &= A(\underbrace{I + At + \frac{A^2}{2!}t^2 + \cdots}_{e^{At}}) = Ae^{At} \end{aligned}$$

10. Primena na homogene sisteme sa konstantnim koeficijentima

$$\frac{dX}{dt} = AX$$

$\Phi(t) = e^{At}$ zadovoljava: $(i)\ \dfrac{d\Phi}{dt} = Ae^{At} = A\Phi$

$(ii)\ det\, e^{At} \neq 0$

$\Rightarrow\ \Phi(t) = e^{\,At}$ je fundamentalna matrica

"Težina" problema nalaženja fundamentalne matrice prebačena je na sumiranje

Košijev problem:

$$X(t_0) = X_0, \quad X = \Phi(t)C = e^{At}C,$$

$$X(t_0) = e^{At_0}C = X_0\ \Rightarrow C = e^{-At_0}X_0\ \Rightarrow$$

Rešenje (KP):

$$\boxed{X = e^{At}e^{-At_0}X_0 = e^{A(t-t_0)}X_0}$$

Primer:

$$\begin{aligned} x_1' &= -2x_2 \\ x_2' &= x_1 - 3x_2 \end{aligned}$$

$$A = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}, \; A^2 = \begin{bmatrix} -2 & 6 \\ -3 & 7 \end{bmatrix}, \; A^3 = \begin{bmatrix} 6 & -14 \\ 7 & -15 \end{bmatrix}$$

$$\Phi(t) = I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots =$$

$$= \begin{bmatrix} 1 - \frac{2t^2}{2!} + \frac{6t^3}{3!} + \cdots & -\frac{2t}{1!} + \frac{6t^2}{2!} - \frac{14t^3}{3!} + \cdots \\ \frac{t}{1!} - \frac{3t^2}{2!} + \frac{7t^3}{3!} + \cdots & 1 - \frac{3t}{1!} + \frac{7t^2}{2!} - \frac{15t^3}{3!} + \cdots \end{bmatrix}$$

Domaći: Naći $\Phi(t) = e^{At}$ za sistem

$$\begin{aligned} x_1' &= 5x_1 + 4x_2 \\ x_2' &= 4x_1 + 5x_2 \end{aligned}$$

11. Izračunavanje matrice e^{At}

$$1^\circ \ A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}, \ e^{At} = \begin{bmatrix} e^{a_1 t} & 0 & 0 \\ 0 & e^{a_2 t} & 0 \\ 0 & 0 & e^{a_3 t} \end{bmatrix}$$

$$2^\circ \ A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}, \qquad e^{At} = \begin{bmatrix} e^{at} & 0 & 0 \\ te^{at} & e^{at} & 0 \\ \frac{t^2}{2!}e^{at} & te^{at} & e^{at} \end{bmatrix}$$

$$3^\circ \ A = \left[\begin{array}{c|cc} a_1 & 0 & 0 \\ \hline 0 & a & 0 \\ 0 & 1 & a \end{array}\right], \ e^{At} = \left[\begin{array}{c|cc} e^{a_1 t} & 0 & 0 \\ \hline 0 & e^{at} & 0 \\ 0 & te^{at} & e^{at} \end{array}\right]$$

Opšti slučaj: Svodjenje matrice A na Žordanov oblik

(knjiga)

Domaći: Proveriti 2°

PARCIJALNE DIFERENCIJALNE JEDNAČINE PRVOG REDA

1. Osnovni pojmovi

$x_1, \ldots, x_n$ - nezavisno promenljive
$u(x_1, \ldots, x_n)$ - nepoznata funkcija

$$(P) \qquad F(x_1, \ldots, x_n, u, \tfrac{\partial u}{\partial x_1}, \ldots, \tfrac{\partial u}{\partial x_n}) = 0$$

Definicija: Funkcija $u(x_1, \ldots, x_n)$ je **rešenje** jednačine (P) na oblasti $D \subset R^n$ ako identički zadovoljava (P) na D.

$$(HL) \quad P_1(x_1, \ldots, x_n)\frac{\partial u}{\partial x_1} + \cdots + P_n(x_1, \ldots, x_n)\frac{\partial u}{\partial x_n} = 0$$

linearna homogena parcijalna jed. prvog reda

$u \equiv$ const je trivijalno rešenje (LH)

$$(KL)\ P_1(x_1, \ldots, x_n, u)\frac{\partial u}{\partial x_1} + \cdots + P_n(x_1, \ldots, x_n, u)\frac{\partial u}{\partial x_n} = \\ = P_{n+1}(x_1, \ldots, x_n, u)$$

kvazilinearna homogena parcijalna jed. prvog reda

2. Linearna homogena parcijalna jednačina prvog reda

Neka su $P_1, \ldots, P_n$ neprekidno diferencijabilne na D i neka je bar jedna od njih, npr. $P_n \neq 0$ na D.

$(HL) \rightarrow$

$$(S) \qquad \frac{dx_1}{P_1(x_1,\ldots,x_n)} = \cdots = \frac{dx_n}{P_n(x_1,\ldots,x_n)}$$

Teorema: Neka je $u = \varphi(x_1, \ldots, x_n)$ neprekidno diferencijabilna funkcija i $u \not\equiv const$. Važi:

φ je rešenje $(LH) \Leftrightarrow \varphi(x_1, \ldots, x_n) = C$ je prvi integral sistema (S)

Dokaz: $P_n \neq 0$ na $D \Rightarrow (S) \Leftrightarrow$

$$\frac{dx_i}{dx_n} = \frac{P_i(x_1, \ldots, x_n)}{P_n(x_1, \ldots, x_n)}, \ i = 1, \ldots, n-1 \quad (x_n \text{ nez. pr.})$$

Neka je $\varphi = C$ prvi integral sistema (S)

$$\Leftrightarrow$$

$$\frac{\partial \varphi}{\partial x_n} + \frac{\partial \varphi}{\partial x_1} \cdot \frac{P_1}{P_n} + \cdots + \frac{\partial \varphi}{\partial x_{n-1}} \cdot \frac{P_{n-1}}{P_n} = 0, \ (x_1, \ldots, x_n) \in D$$

$$\Leftrightarrow$$

$$P_1\frac{\partial\varphi}{\partial x_1}+\cdots+P_n\frac{\partial\varphi}{\partial x_n}=0,\ (x_1,\ldots,x_n)\in D$$

tj. φ je rešenje (HL)

Teorema: Neka su $\varphi_1 = C_1, \ldots, \varphi_k = C_k$ prvi integrali sistema (S), a F proizvoljna neprekidno diferencijabilna funkcija k promenljivih. Tada je

$$u = F(\varphi_1, \ldots, \varphi_k)$$

rešenje (LH) jednačine.

Dokaz:
$$\begin{aligned}\frac{\partial u}{\partial x_1} &= \frac{\partial F}{\partial\varphi_1}\cdot\frac{\partial\varphi_1}{\partial x_1}+\cdots+\frac{\partial F}{\partial\varphi_k}\cdot\frac{\partial\varphi_k}{\partial x_1}\\ &\vdots\\ \frac{\partial u}{\partial x_n} &= \frac{\partial F}{\partial\varphi_1}\cdot\frac{\partial\varphi_1}{\partial x_n}+\cdots+\frac{\partial F}{\partial\varphi_k}\cdot\frac{\partial\varphi_k}{\partial x_n}\end{aligned}$$

$P_1\frac{\partial u}{\partial x_1}+\cdots+P_n\frac{\partial u}{\partial x_n}=$

$$\begin{aligned}P_1\Big(\frac{\partial F}{\partial\varphi_1}\cdot\frac{\partial\varphi_1}{\partial x_1}+&\cdots+\frac{\partial F}{\partial\varphi_k}\cdot\frac{\partial\varphi_k}{\partial x_1}\Big)+\cdots\\ \cdots+P_n\Big(\frac{\partial F}{\partial\varphi_1}\cdot\frac{\partial\varphi_1}{\partial x_n}+&\cdots+\frac{\partial F}{\partial\varphi_k}\cdot\frac{\partial\varphi_k}{\partial x_n}\Big)=\end{aligned}$$

$= \frac{\partial F}{\partial \varphi_1}(P_1\frac{\partial \varphi_1}{\partial x_1} + \cdots + P_n\frac{\partial \varphi_1}{\partial x_n}) + \cdots + \frac{\partial F}{\partial \varphi_k}(P_1\frac{\partial \varphi_k}{\partial x_1} + \cdots + P_n\frac{\partial \varphi_k}{\partial x_n}) = 0$ na D, tj.

$$u = F(\varphi_1, \ldots, \varphi_k) \text{ je rešenje } (HL).$$

Teorema: Ako su $\varphi_1 = C_1, \ldots, \varphi_{n-1} = C_{n-1}$ nezavisni prvi integrali sistema (S), a F proizvoljna neprekidno diferencijabilna funkcija, tada je

$$\boxed{u = F(\varphi_1, \ldots, \varphi_{n-1})}$$

opšte rešenje (HL) jednačine.

Postupak:

1° $(HL) \rightarrow (S)$
2° $\varphi_1 = C_1, \ldots, \varphi_{n-1} = C_{n-1}$ nez. prvi int. sistema (S)
3° $u = F(\varphi_1, \ldots, \varphi_{n-1})$ opšte rešenje (HL)

Napomena: Umesto proizvoljne konstante pojavljuje se proizvoljna funkcija

Primer: $\frac{\partial u}{\partial x}y^2 z - \frac{\partial u}{\partial y}x^2 z + \frac{\partial u}{\partial z}x^2 y = 0$

$$D = \{(x, y, z) : x > 0, y > 0, z > 0\}$$

$$(S) \qquad \frac{dx}{y^2 z} = \frac{dy}{-x^2 z} = \frac{dz}{x^2 y}$$

$$\left.\begin{array}{l} \frac{dx}{y^2 z} = \frac{dy}{-x^2 z} \Rightarrow x^3 + y^3 = C_1 \\ \frac{dy}{-x^2 z} = \frac{dz}{x^2 y} \Rightarrow y^2 + z^2 = C_2 \end{array}\right\} \text{nezavisni su}$$

Opšte rešenje: $u = F(x^3 + y^3, y^2 + z^2)$

3. Problem sa početnim uslovom za (LH)

$$(HL) \qquad P_1 \frac{\partial u}{\partial x_1} + \cdots + P_n \frac{\partial u}{\partial x_n} = 0,$$

P_i neprekidno diferencijabilne funkcije na D, $P_n \neq 0$

Problem: ako $(x_1^0, \ldots, x_n^0) \in D$, naći ono rešenje (LH) jednačine koje za $x_n = x_n^0$ zadovoljava uslov

$$u(x_1, \ldots, x_{n-1}, x_n^0) = \varphi(x_1, \ldots, x_{n-1})$$

gde je φ data funkcija.

Postupak:

1° (S) $\quad \frac{dx_1}{P_1} = \cdots = \frac{dx_n}{P_n}$

2° $$\left.\begin{array}{l} \varphi_1(x_1,\ldots,x_{n-1},x_n) = C_1 \\ \quad\vdots \\ \varphi_{n-1}(x_1,\ldots,x_{n-1},x_n) = C_{n-1} \end{array}\right\} \text{nez. prvi integrali}$$

Neka je zadovoljen dovoljan uslov za nezavisnost:

$$\begin{vmatrix} \frac{\partial \varphi_1}{\partial x_1} & \cdots & \frac{\partial \varphi_1}{\partial x_{n-1}} \\ \vdots & & \vdots \\ \frac{\partial \varphi_{n-1}}{\partial x_1} & \cdots & \frac{\partial \varphi_{n-1}}{\partial x_{n-1}} \end{vmatrix} \neq 0$$

3° $$\left.\begin{array}{l} \varphi_1(x_1,\ldots,x_{n-1},x_n^0) = C_1 \\ \quad\vdots \\ \varphi_{n-1}(x_1,\ldots,x_{n-1},x_n^0) = C_{n-1} \end{array}\right\} \text{jednozn. resiv sis.}$$

4° $$\left.\begin{array}{l} x_1 = \lambda_1(C_1,\ldots,C_{n-1}) \\ \quad\vdots \\ x_{n-1} = \lambda_{n-1}(C_1,\ldots,C_{n-1}) \end{array}\right\} \text{rešenje}$$

Tada je

$$u=\varphi(\lambda_1(C_1,\dots,C_{n-1}),\dots,\lambda_{n-1}(C_1,\dots,C_{n-1}))$$

traženo rešenje, gde konstante $C_1,\dots,C_{n-1}$ treba zameniti na osnovu $\mathbf{2}^\circ$:

$$u=\varphi(\lambda_1(\varphi_1(x_1,\dots,x_n),\dots,\varphi_{n-1}(x_1,\dots,x_n)),\dots \\ \dots,\lambda_{n-1}(\varphi_1(x_1,\dots,,x_n),\dots,\varphi_{n-1}(x_1,\dots,x_n)))$$

Zaista,

1° u jeste rešenje - složena f-ja integrala sistema (S)

2° $u(x_1,\dots,x_{n-1},x_n^0)=$

$$=\varphi[\underbrace{\lambda_1(\varphi_1(x_1,\dots,x_n^0),\dots,\varphi_{n-1}(x_1,\dots,x_n^0))}_{x_1},\dots \\ \dots,\underbrace{\lambda_{n-1}(\varphi_1(x_1,\dots,,x_n^0),\dots,\varphi_{n-1}(x_1,\dots,x_n^0))}_{x_{n-1}}]=$$

$=\varphi(x_1,\dots,x_{n-1})$ (iz 4°)

Primer: Naći ono rešenje u prethodnom zadatku koje za $z=1$ postaje $u(x,y,1)=x+y$, $(y>0)$.

$$
\begin{matrix} x^3 + y^3 = C_1 \\ y^2 + z^2 = C_2 \end{matrix} \rightarrow \begin{matrix} x^3 + y^3 = C_1 \\ y^2 + 1 = C_2 \end{matrix} \rightarrow \begin{matrix} y = \underbrace{\sqrt{C_2 - 1}}_{\lambda_2},\ (y > 0) \\ x = \underbrace{\sqrt[3]{C_1 - (C_2 - 1)^{3/2}}}_{\lambda_1} \end{matrix}
$$

$$
u = \underbrace{\sqrt[3]{C_1 - (C_2 - 1)^{3/2}}}_{x} + \underbrace{\sqrt{C_2 - 1}}_{y} =
$$

$$
= \sqrt[3]{\underbrace{x^3 + y^3}_{C_1} - (\underbrace{y^2 + z^2}_{C_2} - 1)^{3/2}} + \sqrt{\underbrace{y^2 + z^2}_{C_2} - 1}
$$

4. Kvazilinearna jednačina prvog reda

$$
(KL) \quad P_1(x_1, \ldots, x_n, u)\frac{\partial u}{\partial x_1} + \cdots + P_n(x_1, \ldots, x_n, u)\frac{\partial u}{\partial x_n} = \\ = P_{n+1}(x_1, \ldots, x_n, u)
$$

$P_1, \ldots, P_{n+1}$ su nepr. dif. na $\bar{D} \subset R^{n+1}$, $P_n \neq 0$

$(KL) \rightarrow$

$$
(HL) \quad P_1 \frac{\partial v}{\partial x_1} + \cdots + P_n \frac{\partial v}{\partial x_n} + P_{n+1} \frac{\partial v}{\partial u} = 0
$$

Teorema: Neka je funkcija $v = v(x_1, \ldots, x_n, u)$ neprekidno diferencijabilna i $\frac{\partial v}{\partial u} \neq 0$ na $\bar{D}$. Funkcija $v(x_1, \ldots, x_n, u)$ je rešenje (HL) jednačine ako i samo ako je funkcija $u(x_1, \ldots, x_n)$, implicitno zadana sa

$$v(x_1, \ldots, x_n, u) = 0$$

rešenje (KL) jednačine.

Dokaz: $$v(x_1, \ldots, x_n, u) = 0$$

$$\frac{\partial}{\partial x_i}: \quad \frac{\partial v}{\partial x_i} + \frac{\partial v}{\partial u} \cdot \frac{\partial u}{\partial x_i} = 0,\ i = 1, \ldots, n \Rightarrow$$

$$\frac{\partial u}{\partial x_i} = -\frac{\frac{\partial v}{\partial x_i}}{\frac{\partial v}{\partial u}}$$

Neka je $v(x_1, \ldots, x_n, u)$ rešenje (HL), tj.

$$P_1 \frac{\partial v}{\partial x_1} + \cdots + P_n \frac{\partial v}{\partial x_n} + P_{n+1} \frac{\partial v}{\partial u} \equiv 0 \,|\, : (-\frac{\partial v}{\partial u})$$

$$\Leftrightarrow$$

$$P_1 \left(-\frac{\frac{\partial v}{\partial x_1}}{\frac{\partial v}{\partial u}} \right) + \cdots + P_n \left(-\frac{\frac{\partial v}{\partial x_i}}{\frac{\partial v}{\partial u}} \right) - P_{n+1} \equiv 0$$

$$\Leftrightarrow$$

$$P_1 \frac{\partial u}{\partial x_1} + \cdots + P_n \frac{\partial u}{\partial x_n} \equiv P_{n+1}$$

$(HL) \to$

$$(S) \quad \frac{dx_1}{P_1} = \cdots = \frac{dx_n}{P_n} = \frac{du}{P_{n+1}}$$

Neka su su $\varphi_1 = C_1, \ldots, \varphi_n = C_n$ nezavisni prvi integrali sistema (S). Ako je $v = F(\varphi_1, \ldots, \varphi_n)$ opšte rešenje (HL), tada je

$$F(\varphi_1(x_1, \ldots, x_n, u), \ldots \varphi_n(x_1, \ldots, x_n, u)) = 0$$

opšte rešenje (KL).

Postupak: 1° $(KL) \to (S)$

2° $\varphi_1 = C_1, \ldots, \varphi_n = C_n$ nez. prvi int.

3° $F(\varphi_1, \ldots, \varphi_n) = 0$ opšte reš. (KL)

Domaći: Naći opšte rešenje jednačine

$$x(y^2 - z^2)\frac{\partial z}{\partial x} + y(z^2 - x^2)\frac{\partial z}{\partial y} = z(x^2 - y^2)$$

$$(y \neq 0, |z| \neq |x|)$$

5. Problem sa početnim uslovom za (KL) jednačinu

$$(KL) \quad P_1(x_1,\ldots,x_n,u)\frac{\partial u}{\partial x_1}+\cdots+P_n(x_1,\ldots,x_n,u)\frac{\partial u}{\partial x_n} = \\ = P_{n+1}(x_1,\ldots,x_n,u)$$

$P_1,\ldots,P_{n+1}$ su nepr. dif. na $\bar{D}\subset R^{n+1}$, $P_n \neq 0$

Problem: Ako $(x_1^0,\ldots,x_n^0,u^0)\in\bar{D}$, naći ono rešenje (KL) koje za $x_n = x_n^0$ zadovoljava uslov

$$u(x_1,\ldots,x_{n-1},x_n^0) = \varphi(x_1,\ldots,x_{n-1}),$$

gde je φ data funkcija.

Postupak:

1°
$$\frac{dx_1}{P_1}=\cdots=\frac{dx_n}{P_n}=\frac{du}{P_{n+1}} \qquad (S)$$

2°
$$\left.\begin{array}{l}\varphi_1(x_1,\ldots,x_n,u)=C_1\\ \quad\vdots\\ \varphi_n(x_1,\ldots,x_n,u)=C_n\end{array}\right\}\text{ nez. prvi integrali}$$

$$3^\circ \qquad \left.\begin{array}{l} \varphi_1(x_1,\ldots,x_n^0,u) = C_1 \\ \quad\vdots \\ \varphi_n(x_1,\ldots,x_n^0,u) = C_n \end{array}\right\} \text{ jednozn. rešiv sistem}$$

$$4^\circ \qquad \left.\begin{array}{ll} x_1 & = \lambda_1(C_1,\ldots,C_n) \\ \quad\vdots & \\ x_{n-1} & = \lambda_{n-1}(C_1,\ldots,C_n) \\ u & = \lambda_n(C_1,\ldots,C_n) \end{array}\right\} \text{ rešenje}$$

Tada je traženo rešenje:

$$\lambda_n(C_1,\ldots,C_n) = \varphi(\lambda_1(C_1,\ldots,C_n),\ldots,\lambda_{n-1}(C_1,\ldots,C_n))$$

gde $C_1,\ldots,C_n)$ treba na osnovu 2° zameniti sa $\varphi_1,\ldots,\varphi_n$, tj.

$$\lambda_n(\varphi_1(x_1,\ldots,x_n,u),\ldots,\varphi_n(x_1,\ldots,x_n,u)) = \\ \varphi(\lambda_1(\,\cdots\,),\ldots,\lambda_{n-1}(\,\cdots\,))$$

Domaći: Naći ono rešenje jednačine

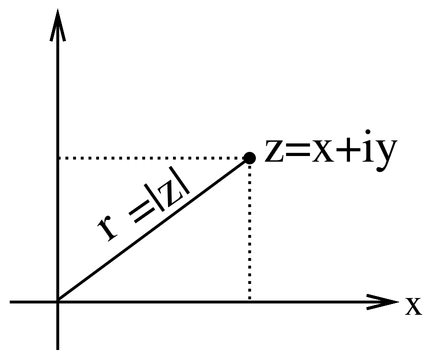
$$x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} + z^2 = 0$$

koje za $y = 1$ postaje $z = x$.

TEORIJA FUNKCIJA KOMPLEKSNE PROMENLJIVE

1. Kompleksni brojevi

- $z = x + iy,\ i^2 = -1$ algebarski oblik
  $x = Re\, z,\ y = Im\, z$



$$|z| = \sqrt{x^2 + y^2}$$
$$\bar{z} = x - iy$$

- $z = \rho(\cos\varphi + i\sin\varphi)$ trigonometrijski oblik

$$\rho = \sqrt{x^2 + y^2} = |z|$$
$$\tan\varphi = \frac{y}{x} \quad (\cos\varphi = x/\rho,\ \sin\varphi = y/\rho)$$
$$arg\, z \stackrel{\text{def}}{=} \varphi,\ 0 \le \varphi < 2\pi\ (-\pi \le \varphi < \pi)$$

$$z = \rho\,(\cos(\varphi + 2k\pi) + i\sin(\varphi + 2k\pi))\,,\ k \in Z$$
$$Arg\, z \stackrel{\text{def}}{=} \varphi + 2k\pi = arg\, z + 2k\pi,\ k \in Z$$

- $z^n = \rho^n(\cos n\varphi + i \sin n\varphi)$

  $$z^{1/n} = \rho^{1/n}\left(\cos\frac{\varphi + 2k\pi}{n} + i\sin\frac{\varphi + 2k\pi}{n}\right),$$
  $$k = 0, \ldots, n-1$$

Primer:

$z = 1 + i; \quad x = 1,\ y = 1, |z| = \sqrt{1+1} = \sqrt{2} = \rho$

$\tan\varphi = \dfrac{1}{1} = 1, (\cos\varphi = \sin\varphi = \dfrac{1}{\sqrt{2}} > 0) \Rightarrow \varphi = \dfrac{\pi}{4}$

$arg\, z = \dfrac{\pi}{4},\ Arg\, z = \dfrac{\pi}{4} + 2k\pi,\ k \in Z$

$z = \sqrt{2}(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}) \Rightarrow z^3 = 2\sqrt{2}(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}),$

$$z^{1/4} = 2^{1/8}(\cos\frac{\pi/4 + 2k\pi}{4} + i\sin\frac{\pi/4 + 2k\pi}{4}),$$
$$k = 0, 1, 2, 3$$

- $z = \rho e^{i\varphi}$ - eksponencijalni oblik

  $e^{i\varphi} = \cos\varphi + i\sin\varphi$ - Ojlerova formula

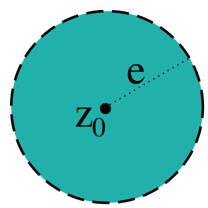
Domaći: Napisati $z = -1$ u trigonometrijskom i eksponencijalnom obliku. Naći sve vrednosti $\sqrt[5]{-1}$.

2. Pojam funkcije kompleksne promenljive

- $\varepsilon -$ okolina: $\quad V_\varepsilon(z_0) = \{z \in C : |z - z_0| < \varepsilon\}$

  $z = x + iy,\ z_0 = x_0 + y_0 \Rightarrow |z - z_0| =$

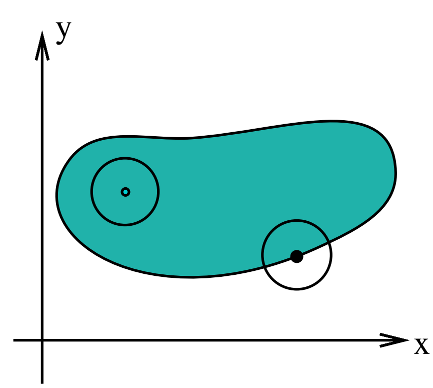
  $$= |(x - x_0) + i(y - y_0)| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$



$$|z - z_0| < \varepsilon \Leftrightarrow (x - x_0)^2 + (y - y_0)^2 < \varepsilon^2$$

- Tačka z_0 je unutrašnja tačka skupa $S \subset C$ ako postoji $\varepsilon > 0$ takvo da je

$$U_\varepsilon(z_0) \subset S.$$



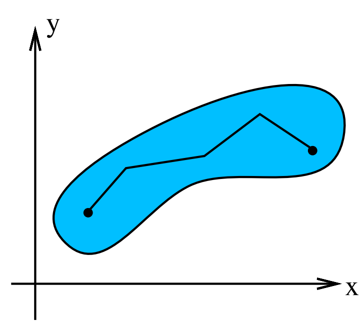
- Tačka z_0 je granična tačka skupa $S \subset C$ ako u svakoj ε-okolini tačke z_0 postoje tačke koje pripadaju S, a takodje i tačke koje ne pripadaju S.

• Skup svih graničnih tačaka skupa S obrazuje granicu skupa S.

• Otvoren skup je skup čija je svaka tačka unutrašnja.

• Zatvoren skup je skup koji sadrži svoju granicu.

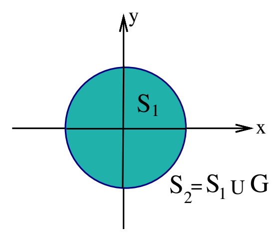
• Povezan skup je skup čije se svake dve tačke mogu povezati poligonalnom linijom koja pripada tom skupu.



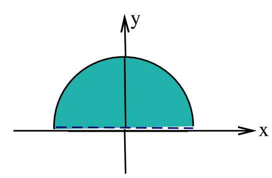
• Oblast je otvoren i povezan skup.

Primeri:

$S_1 = \{z : |z| < 2\}$ - otvoren
$S_2 = \{z : |z| \leq 2\}$ - zatvoren

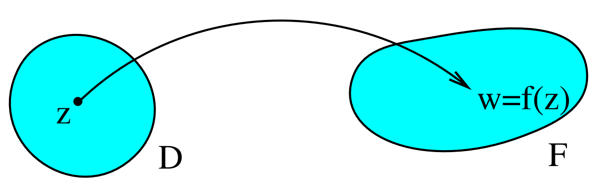


$G = \{z : |z| = 2\}$ - granica za S_1 i S_2 - zatvoren
$S_3 = \{z : |z| \leq 2, Im\, z > 0\}$ - ni otvoren ni zatvoren



Definicija: Funkcija $f : D \to F,\ D, F \subseteq C$ je funkcija kompleksne promenljive.

$$w = f(z),\ z \in D, w \in F$$



Razdvajanjem realnih i imaginarnih delova slika i originala,

$$z = x + iy \quad \longrightarrow \quad w = u + iv$$

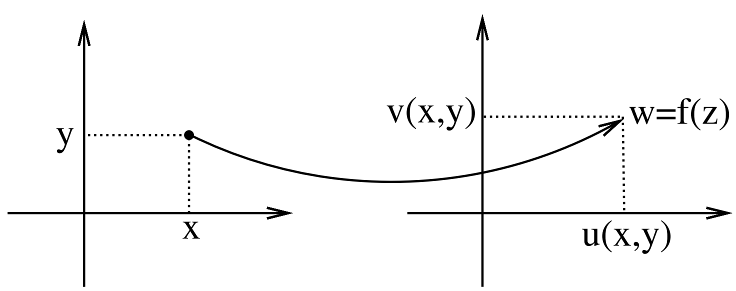
$$u + iv = f(x + iy) \quad \Leftrightarrow \quad u = u(x, y),\ v = v(x, y),$$

dobijaju se dve realne funkcije od dve realne promenljive.

Primer : $w = \bar{z} = x - iy \Rightarrow u = x,\ v = -y$

$$w = \frac{1}{z} = \frac{1}{x + iy} \cdot \frac{x - iy}{x - iy} = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2}$$

$$\Rightarrow u = \frac{x}{x^2 + y^2},\ v = -\frac{y}{x^2 + y^2}$$



- $F = \{w \in C : w = f(z), z \in D\}$
  $f : D \rightarrow F,\ f^{-1} : F \rightarrow D$
  $f^{-1}(w) = \{z \in D : f(z) = w\}$ inverzna funkcija -
  -može biti višeznačna

3. Elementarne funkcije kompleksne promenljive

1° $P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n,\ a_i \in C$ **polinom**

2° $R(z) = \dfrac{P(z)}{Q(z)}$ **racionalna funkcija**; P, Q – polinomi

3° $\boxed{e^z = e^{x+iy} \overset{\text{def}}{=} e^x(\cos y + i \sin y)}$ **eksponencijalna f-ja**

Važi: $e^{z_1} \cdot e^{z_2} = e^{z_1+z_2}; \quad e^{z_1}/e^{z_2} = e^{z_1-z_2}$

4° $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

Važi: $\sin^2 z + \cos^2 z = 1$

$\sin(-z) = -\sin z, \quad \cos(-z) = \cos z$

$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$

$e^{iz} = \cos z + i \sin z,\ (e^{i\varphi} = \cos\varphi + i \sin\varphi)$

5° $\sqrt[n]{z} = \sqrt[n]{|z|}(\cos \dfrac{arg\, z + 2k\pi}{n} + i \sin \dfrac{arg\, z + 2k\pi}{n}),$

$k = 0, \ldots, n-1$

Višeznačna funkcija, n grana, inverzna za $z = w^n$

6° $w = Ln\, z$ - inverzna funkcija za $z = e^w, z \neq 0$

$$z = |z|e^{i\,arg\,z}\,, w = u + iv \Rightarrow$$
$$\underbrace{|z|e^{i\,arg\,z}}_{z} = e^{u+iv} = \underbrace{e^{u}e^{iv}}_{e^{w}} \Rightarrow$$
$$\left.\begin{array}{l} e^{u} = |z|, \text{ tj. } u = \ln|z| \\ v = arg\,z + 2k\pi = Arg\,z \end{array}\right\}$$

$w = Ln\,z = \ln|z| + i\,Arg\,z$ -višeznačna funkcija

-beskonačno mnogo grana

$\ln z = \ln|z| + i\,arg\,z, \quad z \neq 0, \quad 0 \leq arg\,z < 2\pi$

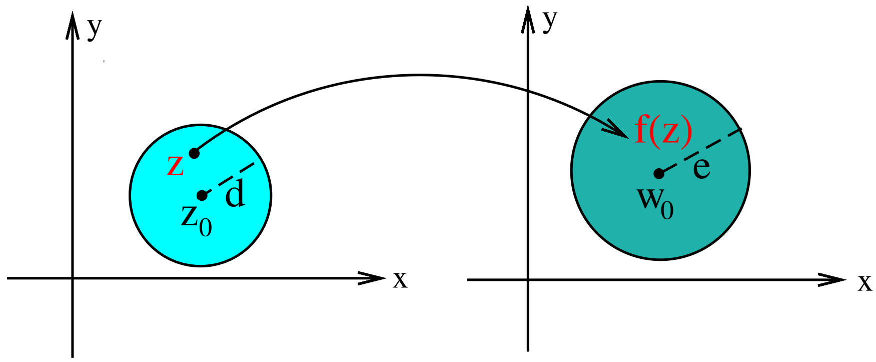
(glavna grana) $\quad (-\pi \leq arg\,z < \pi)$

Domaći: Izračunati e^{1-i}, $\sin(1-i)$, $Ln\,i$, $\ln i$ u algebarskom obliku.

3. Granična vrednost i neprekidnost

Definicija: $w_0 = \lim\limits_{z \to z_0} f(z) \Leftrightarrow$

$$(\forall \varepsilon > 0)(\exists \delta(\varepsilon) > 0)(0 < |z - z_0| < \delta \Rightarrow |f(z) - w_0| < \varepsilon)$$



Teorema: Neka je $z = x + iy$, $z_0 = x_0 + iy_0$, $w_0 = u_0 + iv_0$ i $f(z) = u(x, y) + iv(x, y)$. Tada:

$$\lim_{z \longrightarrow z_0} f(z) = w_0 \quad \Leftrightarrow$$

$$\lim_{(x,y)\longrightarrow(x_0,y_0)} u(x, y) = u_0,$$
$$\lim_{(x,y)\longrightarrow(x_0,y_0)} v(x, y) = v_0.$$

Primer:

$$\lim_{z\longrightarrow 0} e^z = \lim_{x+iy\longrightarrow 0}(e^x \cos y + ie^x \sin y) = \lim_{(x,y)\longrightarrow(0,0)} e^x \cos y + \\ + i \lim_{(x,y)\longrightarrow(0,0)} e^x \sin y = 1.$$

Definicija: Funkcija f je neprekidna u tački z_0 $\Leftrightarrow$

$$\boxed{\lim_{z\longrightarrow z_0} = f(z_0)}$$

Teorema: Funkcija $f(z) = f(x+iy) = u(x,y) + +iv(x,y)$ je neprekidna u tački z_0 ako i samo ako su funkcije $u(x,y)$ i $v(x,y)$ neprekidne u tački (x_0, y_0).

Teorema: Ako su f_1 i f_2 neprekidne funkcije, onda su

$$f_1 \pm f_2,\ f_1 \cdot f_2,\ f_1/f_2\ (f_2 \neq 0)$$

takodje neprekidne funkcije.

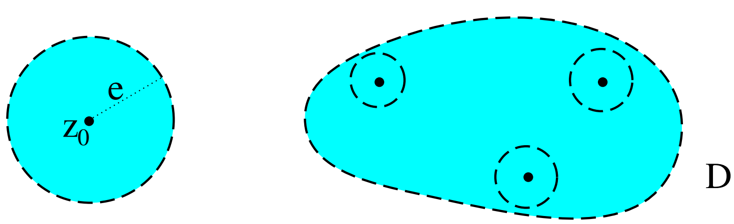
5. Izvod funkcije kompleksne promenljive

Definicija:

$$\boxed{f'(z_0) = \lim_{\triangle z \to 0} \frac{f(z_0+\triangle z)-f(z_0)}{\triangle z}}$$

Definicija: Funkcija f je diferencijabilna u tački z_0 ako postoji $f'(z_0)$.

Definicija: Funkcija f je analitička u tački z_0 ako postoji $\varepsilon > 0$ takvo da je f diferencijabilna $\forall z \in U_\varepsilon(z_0)$.



Definicija: Funkcija f analiticka na oblasti D ako je diferencijabilna u svakoj tački te oblasti.

Primer: $f(z) = z^n, \ n \in N$

$$\lim_{\triangle z \to 0} \frac{(z_0 + \triangle z)^n - z^n}{\triangle z} = \lim_{\triangle z \to 0} \frac{z^n + \binom{n}{1} z^{n-1} \triangle z + \cdots + (\triangle z)^n - z^n}{\triangle z} =$$
$$= n z^{n-1}$$

$$(z^n)' = n z^{n-1} \ \Rightarrow \ z^n \text{ je analitička na } C$$

Teorema: Ako su f i g analitičke na D, onda je:

1° $(f(z) \pm g(z))' = f'(z) \pm g'(z)$

2° $(cf(z))' = cf'(z)$

3° $(f(z)g(z))' = f'(z)g(z) + f(z)g'(z)$

4° $\left(\frac{f(z)}{g(z)}\right)' = \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}, \ g(z) \neq 0$

5° $f(g(z))' = f'_g(g(z))g'_z(z)$

6° $(f(z))' = \frac{1}{(f^{-1}(w))'}$, ako je f^{-1} jednozn. f-ja

Teorema (neophodni uslovi diferencijabilnosti): Ako je $f(z) = u(x,y) + iv(x,y)$ diferencijabilna u tački $z_0 = x_0 + iy_0$, tada postoje parcijalni izvodi

$$\frac{\partial u}{\partial x}(x_0,y_0),\ \frac{\partial u}{\partial y}(x_0,y_0),\ \frac{\partial v}{\partial x}(x_0,y_0),\ \frac{\partial v}{\partial y}(x_0,y_0)$$

i pri tome važe Koši-Rimanovi uslovi:

$$(KR)\ \boxed{\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0),\ \frac{\partial u}{\partial y}(x_0,y_0) = -\frac{\partial v}{\partial x}(x_0,y_0)}$$

Dokaz: $f'(z_0) = \lim_{\triangle z \to 0} \frac{f(z_0+\triangle z) - f(z_0)}{\triangle z} =$

$$\lim_{\triangle x + i\triangle y \to 0} \frac{u(x_0+\triangle x, y_0+\triangle y) + iv(x_0+\triangle x, y_0+\triangle y) - u(x_0,y_0) - iv(x_0,y_0)}{\triangle x + i\triangle y}$$

Specijalno, za $\triangle x = 0$:

$$f'(z_0) = \lim_{\triangle y \to 0} \left[\frac{u(x_0,y_0+\triangle y) - u(x_0,y_0)}{i\triangle y} + i\frac{v(x_0,y_0+\triangle y) - v(x_0,y_0)}{i\triangle y}\right]$$
$$= \frac{\partial v}{\partial y}(x_0,y_0) - i\frac{\partial u}{\partial y}(x_0,y_0). \qquad (\tfrac{1}{i} = -i)$$

Takodje, za $\triangle y = 0$ imamo:

$$f'(z_0) = \lim_{\triangle x \to 0} \left[\frac{u(x_0+\triangle x, y_0) - u(x_0, y_0)}{\triangle x} + i\frac{v(x_0+\triangle x, y_0) - v(x_0, y_0)}{\triangle x} \right]$$
$$= \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0).$$

Izjednačavanjem realnih i imaginarnih delova u dobijenim izrazima za $f'(z_0)$ dobijamo (KR) uslove.

Primer: $f(z) = \bar{z} = x - iy$

$$u = x,\ v = -y; \quad \frac{\partial u}{\partial x} = 1,\ \frac{\partial v}{\partial y} = -1$$

(KR) uslovi nisu zadovoljeni ni za jedno z, pa funkcija nije diferencijabilna ni za jedno z.

Teorema:(dovoljni uslovi za diferencijabilnost): Ako su funkcije $u(x, y)$ i $v(x, y)$ diferencijabilne u tački (x_0, y_0) i ako su u toj tački zadovoljeni (KR) uslovi, onda je funkcija $f(z)$ diferencijabilna u tački $z_0 = x_0 + iy_0$.

Dokaz:

$$u(x_0 + \triangle x, y_0 + \triangle y) - u(x_0, y_0) =$$
$$= \frac{\partial u}{\partial x}(x_0, y_0)\triangle x + \frac{\partial u}{\partial y}(x_0, y_0)\triangle y + \alpha_1 \triangle x + \beta_1 \triangle y,$$
$$(\alpha_1 \to 0, \beta_1 \to 0)$$

$$v(x_0+\triangle x, y_0+\triangle y)-v(x_0,y_0)=$$
$$=\frac{\partial v}{\partial x}(x_0,y_0)\triangle x+\frac{\partial v}{\partial y}(x_0,y_0)\triangle y+\alpha_2\triangle x+\beta_2\triangle y,$$
$$(\alpha_2\to 0, \beta_2\to 0)$$
$$\frac{f(z_0+\triangle z)-f(z_0)}{\triangle z}=$$

$$=\frac{\frac{\partial u}{\partial x}\triangle x+\frac{\partial u}{\partial y}\triangle y+\alpha_1\triangle x+\beta_1\triangle y}{\triangle x+i\triangle y}+i\frac{\frac{\partial v}{\partial x}\triangle x+\frac{\partial v}{\partial y}\triangle y+\alpha_2\triangle x+\beta_2\triangle y}{\triangle x+i\triangle y}=$$
$$=\frac{\frac{\partial u}{\partial x}\triangle x-\frac{\partial v}{\partial x}\triangle y+\alpha_1\triangle x+\beta_1\triangle y}{\triangle x+i\triangle y}+i\frac{\frac{\partial v}{\partial x}\triangle x+\frac{\partial u}{\partial x}\triangle y+\alpha_2\triangle x+\beta_2\triangle y}{\triangle x+i\triangle y}=$$
$$=\frac{\partial u}{\partial x}\cdot\frac{\triangle x+i\triangle y}{\triangle x+i\triangle y}+i\frac{\partial v}{\partial x}\cdot\frac{\triangle x-1/i\triangle y}{\triangle x+i\triangle y}+\underbrace{\frac{\alpha_1\triangle x+\beta_1\triangle y}{\triangle x+i\triangle y}}_{\to 0}+$$

$$+i\underbrace{\frac{\alpha_2\triangle x+\beta_2\triangle y}{\triangle x+i\triangle y}}_{\to 0},$$ $\triangle x\to 0, \triangle y\to 0$. Obzirom da je $-\frac{1}{i}=i$, konačno se dobija

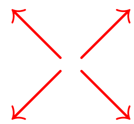
$$\lim_{\triangle z\to 0}\frac{f(z_0+\triangle z)-f(z_0)}{\triangle z}=\frac{\partial u}{\partial x}(x_0,y_0)+i\frac{\partial v}{\partial x}(x_0,y_0)$$

Napomena: Iz prethodnog slede formule

$$\begin{aligned} f'(z_0) &= \frac{\partial u}{\partial x}(x_0,y_0)+i\frac{\partial v}{\partial x}(x_0,y_0)\\ &= \frac{\partial v}{\partial y}(x_0,y_0)-i\frac{\partial u}{\partial y}(x_0,y_0)\end{aligned}$$

Primer: $f(z) = e^z = e^x(\cos y + i \sin y) \Rightarrow$
$u = e^x \cos y, v = e^x \sin y$ (diferencijabilne na R^2)

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial v}{\partial x} = e^x \sin y$$
$$\frac{\partial u}{\partial y} = -e^x \sin y \qquad \frac{\partial v}{\partial y} = e^x \cos y$$



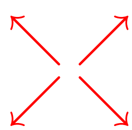
(KR) uslovi važe na R^2 pa je e^z diferencijabilna funkcija i

$$\begin{aligned}(e^z)' &= \frac{\partial}{\partial x}(e^x \cos y) + i\frac{\partial}{\partial x}(e^x \sin y) = \\ &= e^x \cos y + ie^x \sin y = e^x(\cos y + i \sin y) = e^z\end{aligned}$$

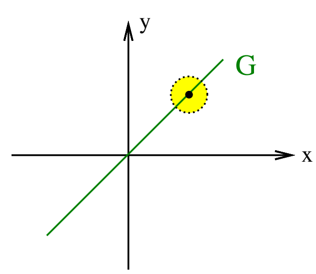
Štaviše, C je otvoren skup, pa je e^z analitička na C

Primer: $f(z) = x^4 + iy^4 \Rightarrow u = v = x^4$, dif. na R^2

$$\frac{\partial u}{\partial x} = 4x^3 \qquad \frac{\partial v}{\partial x} = 0$$
$$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial y} = 4y^3$$



(KR) uslovi važe za $x = y$,



$f(z)$ je diferencijabilna na $G = \{x+iy : x = y\}$, $f'(z) = 4x^3$, ali $f(z)$ nije analitička ni za jedno z.

Domaći: Ispitati diferencijabilnost i analitičnost funkcije

$$f(z) = \bar{z}z^2.$$

6. Primena Koši-Rimanovih uslova

1° Ako je $f(z) = u(x, y)+iv(x, y)$ analitička funkcija, tada u i v zadovoljavaju uslove

$$(1) \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (2) \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Dokaz: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow$

$$\Rightarrow \begin{cases} \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y \partial x}, \ \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y} \Rightarrow \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \Rightarrow (1) \\ \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2}, \ \frac{\partial^2 u}{\partial y \partial x} = -\frac{\partial^2 v}{\partial x^2} \Rightarrow \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 v}{\partial x^2} \Rightarrow (2) \end{cases}$$

2° Ako je data diferencijabilna funkcija $u(x, y)$ tako da važi (1) i ako je $f(z) = u(x, y)+iv(x, y)$ analitička funkcija, onda se funkcija $v(x,y)$ može odrediti do na nepoznatu konstantu.

Treba naći v tako da je

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

tj. funkciju čiji je totalni diferencijal poznat:

$$(\underbrace{-\frac{\partial u}{\partial y}}_{P})dx + (\underbrace{\frac{\partial u}{\partial x}}_{Q})dy \qquad (\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \text{ zbog (1)})$$

Primer: $u(x,y) = x^2 - y^2 - x$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$, važi (1);
$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y \Rightarrow v = \int 2y\,dx + \varphi(y) = 2xy + \varphi(y)$
$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} :\ 2x + \varphi'(y) = 2x - 1 \Rightarrow \varphi(y) = -y + C$

$$\Rightarrow v(x,y) = 2xy - y + C$$

$$f(z) = x^2 - y^2 - x + i(2xy - y + C) = z^2 - z + iC, \ (C \in R)$$

7. Izvodi elementarnih funkcija

1° $f(z) = z^n,\ f'(z) = nz^{n-1)},$ analitička na C

2° $f(z) = \frac{1}{z},\ f'(z) = -\frac{1}{z^2},$ anal. na $D = \{z : z \neq 0\}$

3° $f(z) = e^z,\ f'(z) = e^z,$ analitička na C

4° $f(z) = \sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

$$f'(z) = \frac{ie^{iz} + ie^{-iz}}{2i} = \frac{e^{iz} + e^{-iz}}{2} = \cos z,$$ anal. na C

5° $f(z) = \cos z,\ f'(z) = -\sin z,$ analitička na C

6° $f(z) = \ln z,\ w = \ln z \Leftrightarrow z = e^w$

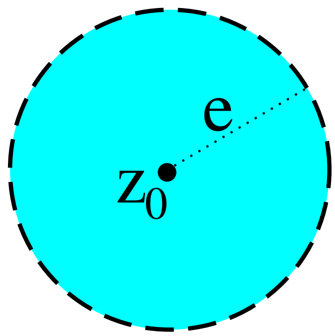
$$w'_z = \frac{1}{z'_w} = \frac{1}{(e^w)'} = \frac{1}{e^w} = \frac{1}{z},$$ analitička na

$D = \{z : -\pi < arg z < \pi\}$

8. Singularne tačke

Definicija: Tačke u kojima funkcija $f(z)$ nije analitička su singularne tačke.

Definicija: Singularna tačka z_0 je izolovani singularitet ako postoji ε-okolina $U_\varepsilon(z_0)$ tačke z_0 koja ne sadrži druge singularitete osim z_0.



$$f \text{ je analitička na } U_\varepsilon(z_0)/\{z_0\}$$

Primeri: 1. $f(z) = \frac{1}{z}$ - ima izolovani sing. $z_1 = 0$

2. $f(z) = \bar{z}$ - je singularna na C, nema izolovanih singulariteta

Tipovi izolovanih singulariteta:

1° otklonjivi singularitet
2° pol
3° esencijalni singularitet

Definicija: Izolovani singularitet z_0 je otklonjivi singularitet ako postoji

$$\lim_{z \to z_0} f(z) = w_0.$$

Pokazuje se da je tada funkcija

$$f_1(z) = \begin{cases} f(z), z \neq z_0 \\ w_0, \quad z = z_0 \end{cases}$$

analitička u z_0.

Primer: Važi: $\lim\limits_{z \longrightarrow 0} \frac{\sin z}{z} = 1$ (i ovde važi Lopitalova teorema). Sledi da je $z = 0$ otklonjivi singularitet. Funkcija

$$f_1(z) = \begin{cases} \frac{\sin z}{z}, z \neq 0 \\ 1, \quad z = 0 \end{cases}$$

je analitička na C.

Definicija: Izolovani singularitet z_0 je pol ako je

$$\lim_{z \longrightarrow z_0} f(z) = \infty$$

Napomena: $\lim\limits_{z \longrightarrow z_0} f(z) = \infty \Leftrightarrow \lim\limits_{z \longrightarrow z_0} |f(z)| = \infty$

Specijalno, pol z_0 je pol reda n ako je

$$\lim_{z \longrightarrow z_0} (z - z_0)^n f(z) = w_0$$

Primer: $f(z) = \frac{1}{(z-1)^2}$ ima pol reda 2 u $z = 1$:

$$\lim_{z \to 1} (z-1)^2 \cdot \frac{1}{(z-1)^2} = 1$$

Definicija: Izolovani singularitet z_0 je esencijalni singularitet ako ne postoji granična vrednost $\lim_{z \to z_0} f(z)$ (konačna ili beskonačna).

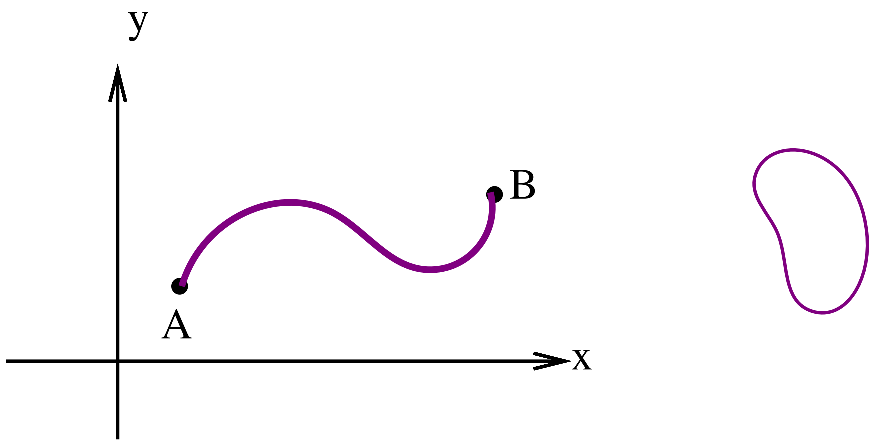
Primer: Funkcija $f(z) = e^{1/z}$ ima esencijalni singularitet u $z = 0$.

9. Integral funkcije kompleksne promenljive

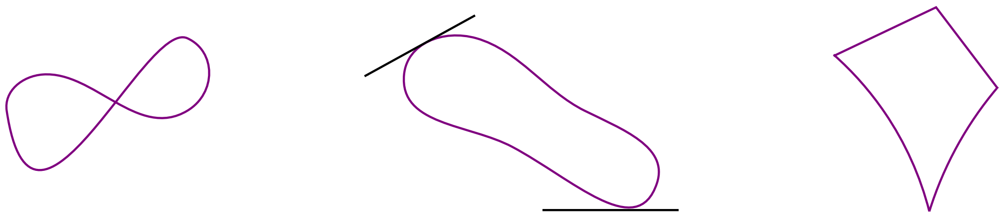
- $x(t), y(t)$ - neprekidne funkcije, $a \leq t \leq b$

$$C : \; z(t) = x(t) + iy(t)$$

je neprekidna kriva koja spaja tačke $A = z(a), b = z(b)$.



- Zatvorena kriva: $A = B$
- Prosta zatvorena kriva: ne seče sebe

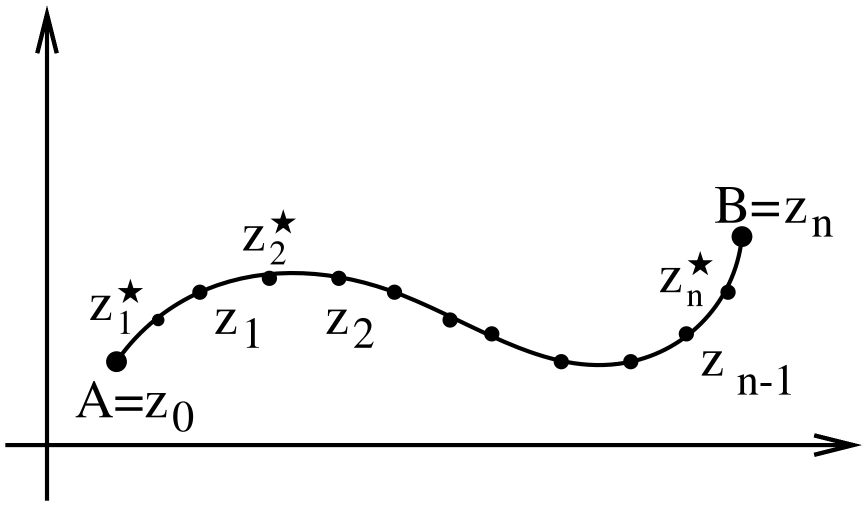


- Glatka kriva: $x(t), y(t)$ su diferencijabilne funkcije

$$z'(t) = x'(t) + iy'(t)$$

- Deo po deo glatka kriva: Sastoji se od konačno mnogo glatkih delova

- Kontura: prosta, zatvorena, deo po deo glatka kriva



$a = t_0 < t_1 < \cdots < t_n = b$
$z_0 = A, z_1 = z(t_1), \ldots, z_n = z(t_n) = B$
$z_k^* \in z_{\widehat{k-1}z_k}$

$$S_n = \sum_{k=1}^{n} f(z_k^*) \triangle z_k, \quad \triangle z_k = z_k - z_{k-1}$$

Definicija: Ako je f neprekidna na $\widehat{AB}$, onda je

$$\boxed{\int_{\widehat{AB}} f(z)\, dz = \lim_{\substack{n \to \infty \\ \max|\triangle z_k| \to 0}} \sum_{k=1}^{n} f(z_k^*) \triangle z_k}$$

Napomena: $\int_{\widehat{AB}} = \int_C$

$z_k = x_k + iy_k,\ z_k^* = x_k^* + iy_k^*$
$\triangle x_k = x_k - x_{k-1},\ \triangle y_k = y_k - y_{k-1} \Rightarrow$

$$\triangle z_k = \triangle x_k + i\triangle y_k$$

$$\begin{aligned} S_n &= \sum_{k=1}^{n} (u(x_k^*, y_k^*) + iv(x_k^*, y_k^*))(\triangle x_k + i\triangle y_k) = \\ &= \sum_{k=1}^{n} (u(x_k^*, y_k^*)\triangle x_k - v(x_k^*, y_k^*)\triangle y_k) + \\ &+ i\sum_{k=1}^{n} (v(x_k^*, y_k^*)\triangle x_k + u(x_k^*, y_k^*)\triangle y_k) \Rightarrow \end{aligned}$$

$$\int_C f(z)\, dz = \underbrace{\int_C udx - vdy + i\int_C vdx + udy}_{\text{krivolinjski integrali druge vrste}}$$

Napomena: Ako je C : $\begin{array}{l} x = x(t) \\ y = y(t), a \leq t \leq b \end{array}$ onda je

$$\int_C P(x,y)dx + Q(x,y)dy =$$

$$= \int_a^b P(x(t),y(t))x'(t)dt + \int_a^b Q(x(t),y(t))y'(t)dt$$

Osobine: 1° $\displaystyle\int_C cf(z)\,dz = c\int_C f(z)\,dz$

2° $\displaystyle\int_C (f(z)+g(z))dz = \int_C f(z)dz + \int_C g(z)dz$

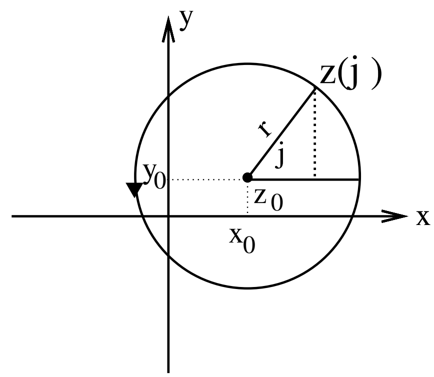
3° $\displaystyle\int_{AB} f(z)dz = -\int_{BA} f(z)dz$

4° $\displaystyle\int_{C_1+C_2} = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$

5° $\displaystyle\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt,\ C : z = z(t),$

$a \le t \le b$

Primer: $\int\limits_C \frac{dz}{z-z_0}$, C : kružnica pol. ρ sa centrom u z_0



Parametarska jednačina kružnice:

$$z(\varphi) = x(\varphi) + iy(\varphi)$$

$$x(\varphi) = x_0 + \rho \cos \varphi$$
$$y(\varphi) = y_0 + \rho \sin \varphi, \ 0 \leq \varphi \leq 2\pi$$

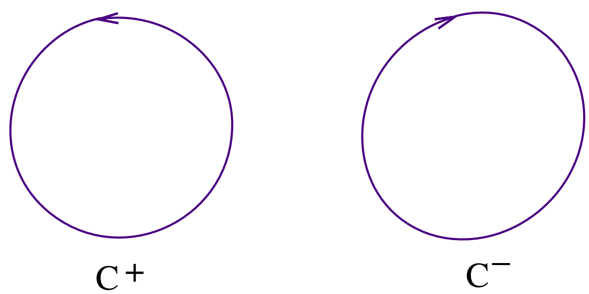
$$\begin{aligned} z(\varphi) &= x_0 + \rho \cos \varphi + i(y_0 + \rho \sin \varphi) = \\ &= x_0 + iy_0 + \rho(\cos \varphi + i \sin \varphi) = \\ &= z_0 + \rho e^{i\varphi}, \quad 0 \leq \varphi \leq 2\pi \end{aligned}$$

$dz = \rho i e^{i\varphi} d\varphi$

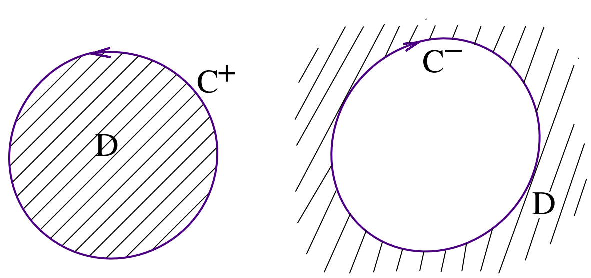
$$\int\limits_C \frac{dz}{z-z_0} = \int\limits_0^{2\pi} \frac{\rho i e^{i\varphi}}{\rho e^{i\varphi}} d\varphi = i \int\limits_0^{2\pi} d\varphi = 2\pi i$$

10. Košijeva teorema

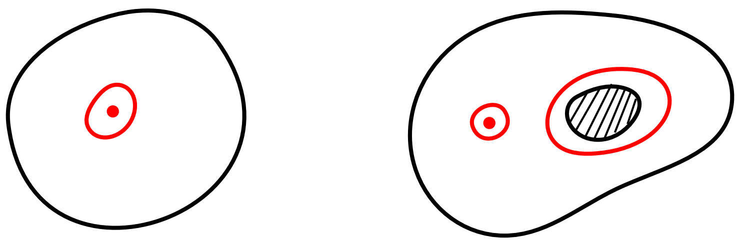
• **Pozitivan obilazak konture:** suprotan kretanju kazaljke na satu



• **Pozitivan obilazak granice oblasti** D: oblast ostaje s leve strane



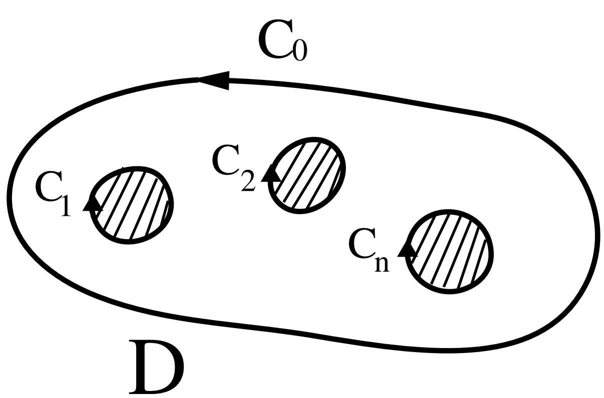
• **Jednostruko povezana oblast:** Svaka kontura se može deformisati u tačku bez napuštanja oblasti $\longrightarrow$



jednostruko povezana

nije jednostruko povezana

• **Višestruko povezana oblast:** Ograničena spolja sa C_0, iznutra sa $C_1, \ldots, C_n$



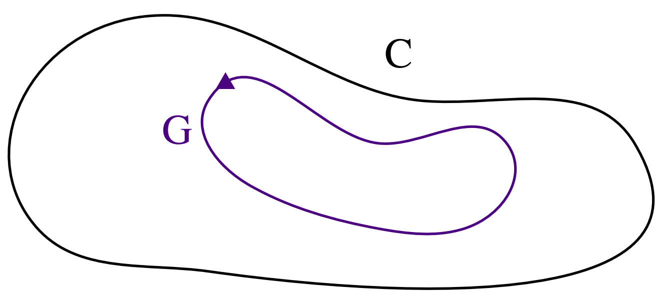
Pozitivan obilazak granice oblasti D : $C_0^+, C_1^-, \ldots, C_n^-$

Košijeva teorema (za jednostruko povezanu oblast):
Ako je funkcija $f(z)$ analitička na jednostruko povezanoj oblasti D i njenoj granici C, tada je

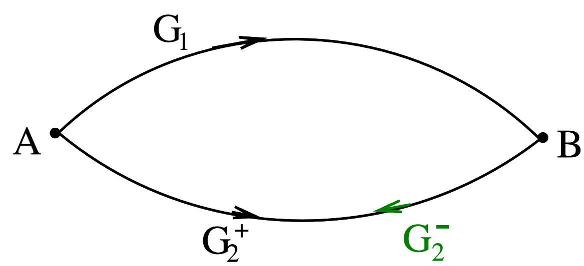
$$\boxed{\int_{C^+} f(z)dz = 0}$$

Dokaz: literatura

Posledica 1: $\int_{G^+} f(z)dz = 0$, G - kontura u D



Posledica 2: $\int_{AB} f(z)dz$ je nezavisan od puta koji spaja tačke A i B $\longrightarrow$



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Dokaz: $C: \ G_1 + G_2^-$ - je kontura, pa je $\int\limits_C f(z)dz = 0$, tj.

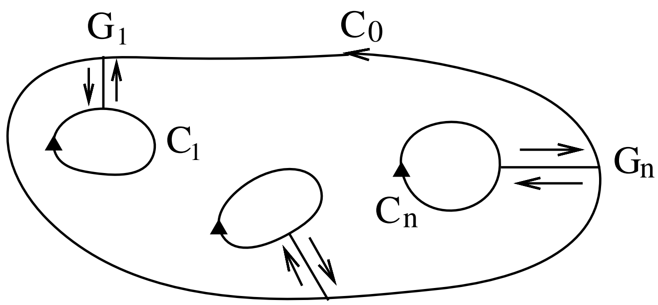
$$\int\limits_{G_1+G_2^-} f(z)dz = \int\limits_{G_1} f(z)dz + \int\limits_{G_2^-} f(z)dz = 0 \Rightarrow$$

$$\int\limits_{G_1} f(z)dz = -\int\limits_{G_2^-} f(z)dz = \int\limits_{G_2} f(z)dz$$

Košijeva teorema (za višestruko povezanu oblast): Neka je višestruko povezana oblast D ograničena spolja konturom C_0, a iznutra sa $C_1, \ldots, C_n$. Ako je $f(z)$

analitička na D i $C = C_0 + \cdots + C_n$, tada je

$$\int\limits_{C^+} f(z)dz = 0$$



Dokaz: Ako se dodaju zaseci $G_1, \ldots, G_n$, dobija se jednostruko povezana oblast $\bar{D}$ sa granicom $\bar{C}$. Obzirom da se zaseci obilaze dva puta u suprotnim smerovima, to je $\int\limits_{\bar{C}^+} f(z)dz = 0$, tj.

$$0 = \int\limits_{C_0^+} + \int\limits_{G_1^+} + \int\limits_{G_1^-} + \cdots + \int\limits_{G_n^+} + \int\limits_{G_n^-} + \int\limits_{C_1^-} + \cdots + \int\limits_{C_n^-} =$$

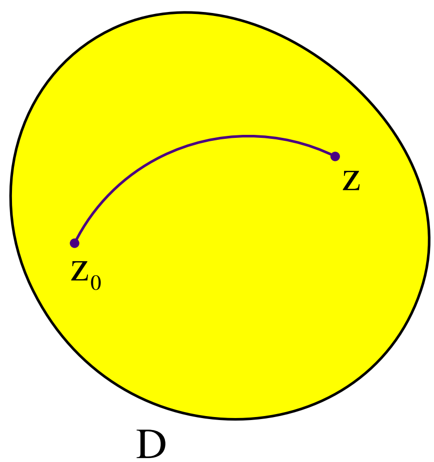
$$= \int\limits_{C_0^+} + \int\limits_{C_1^-} + \cdots + \int\limits_{C_n^-} = \int\limits_{C^+}$$

Posledica:

$$\int_{C_0^+} = \int_{C_1^+} + \cdots + \int_{C_n^+}$$

11. Neodredjeni integral

Neka je funkcija $f(z)$ analitička na jednostruko povezanoj oblasti D i neka z_0, $z \in D$.



Tada integral $\int\limits_{\widehat{z_0 z}} f(z)dz$ ne zavisi od puta, pa možemo pisati

$$F(z) = \int_{z_0}^{z} f(z)dz$$

Teorema: Funkcija $F(z)$ je analitička i pri tome je

$$F'(z) = f(z), \ z \in D$$

Definicija: Primitivna funkcija funkcije $f(z)$ je svaka funkcija sa svojstvom $F'(z) = f(z)$

Definicija: Neodredjeni integral $\int f(z)dz$ je skup svih primitivnih funkcija funkcije $f(z)$.

Primeri:

$$\int z^n dz = \frac{z^{n+1}}{n+1} + C$$

$$\int e^z dz = e^z + C$$

$$\int \cos z dz = \sin z + C$$

$$\int \sin z = -\cos z + C$$

$$\int \frac{dz}{z} = \ln z + C$$

Osobine:

1° $\displaystyle\int (f_1(z) \pm f_2(z))dz = \int f_1(z)dz + \int f_2(z)dz$

2° $\displaystyle\int cf(z)dz = c\int f(z)dz$

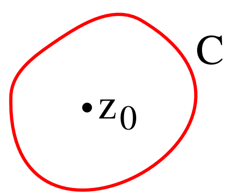
3° $\displaystyle\int\limits_{z_1}^{z_2} f(z)dz = F(z)|_{z_1}^{z_2} = F(z_2) - F(z_1)$

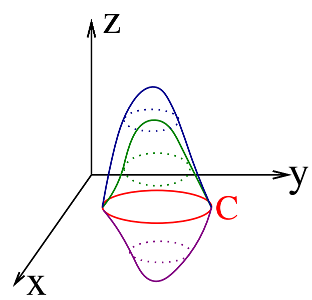
(Njutn-Lajbnicova formula)

Primer: $\int\limits_{1+i}^{1-i} \sin z dz = -\cos z|_{1+i}^{1-i} = \cos(1+i) - $
$-\cos(1-i) = \cos 1(e + e^{-1}) = 2\cos 1\, ch1$

12. Košijeve formule

Ako je poznata vrednost analitičke funkcije na konturi oko z_0, onda je vrednost funkcije u samoj tački z_0 jednoznačno odredjena. $\longrightarrow$





Osobina ne vazi za funkcije dve promenljive

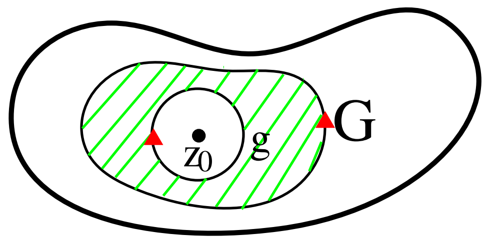
Teorema: Ako je $f(z)$ analitička funkcija na jednostruko povezanoj oblasti D i ako je $G \subset D$ kontura oko $z_0 \in D$, onda je

$$\boxed{f(z_0) = \frac{1}{2\pi i} \int\limits_{G^+} \frac{f(z)dz}{z-z_0} dz}$$

Dokaz: Neka je:

γ - krug radijusa ρ oko z_0, $\gamma \subset D'$;

D' - višestruko povezana oblast ograničena spolja sa G, a iznutra sa γ.



Tada je funkcija $\frac{f(z)}{z-z_0}$ analitička na D' i $G+\gamma$, pa je, prema posledici Košijeve teoreme za višestruko povezane oblasti,

$$\int\limits_{G^+} \frac{f(z)dz}{z-z_0}dz = \int\limits_{\gamma^+} \frac{f(z)dz}{z-z_0}dz = \begin{pmatrix} z = z_0 + \rho e^{i}\varphi \\ o \leq \varphi \leq 2\pi \\ dz = \rho i e^{i\varphi} d\varphi \end{pmatrix}$$

$$= i \int\limits_0^{2\pi} \frac{f(z_0+\rho e^{i\varphi})\rho e^{i\varphi}}{\rho e^{i\varphi}} d\varphi \ \Rightarrow$$

$$\underbrace{\lim_{\rho\to 0} \int\limits_{G^+} \frac{f(z)dz}{z-z_0}dz}_{\text{ne zavisi od } \rho} = i \lim_{\rho\to 0} \int\limits_0^{2\pi} f(z_0+\rho e^{i\varphi})d\varphi \Rightarrow$$

$$\Rightarrow \int\limits_{G^+} \frac{f(z)dz}{z - z_0} dz = if(z_0) \int\limits_0^{2\pi} d\varphi = 2\pi i f(z_0).$$

Posledica: Neka je funkcija $f(z)$ analitička na jednostruko povezanoj oblasti D ograničenoj konturom C i na konturi C. Tada $\forall z_0 \in D$:

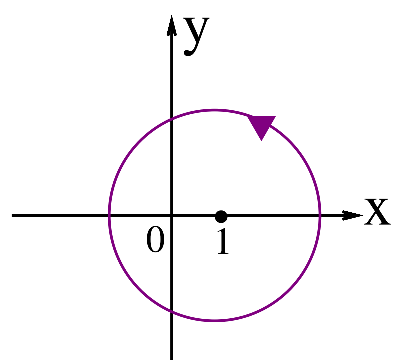
$$\boxed{f(z_0) = \frac{1}{2\pi i} \int\limits_{C^+} \frac{f(z)dz}{z-z_0} dz} \quad \textbf{prva formula}$$

Teorema: Neka je funkcija $f(z)$ analitička na jednostruko povezanoj oblasti D ograničenoj konturom C i na konturi C. Tada na D postoje svi izvodi funkcije f, $\forall z_0 \in D$:

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i} \int\limits_{C^+} \frac{f(z)dz}{(z-z_0)^{n+1}} dz} \quad \textbf{druga formula}$$

Košijeve formule se mogu primeniti na računanje integrala.

Primer: $\int\limits_{C^+} \frac{e^z}{(z-1)^3} dz, \quad C: |z-1| = 3$



$f(z) = e^z$ je analitička, $n + 1 = 3 \Rightarrow n = 2, z_0 = 1$

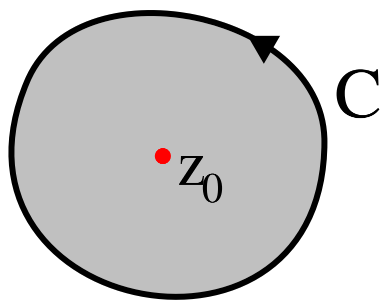
$$\int\limits_{C^+} \frac{e^z}{(z-1)^3} dz = \frac{2\pi i}{2!} f''(1) = \pi i e$$

13. Reziduum funkcije kompleksne promenljive

Definicija: Neka je $z = z_0$ izolovani singularitet funkcije $f(z)$. Tada je

$$Res[f(z), z_0] \stackrel{\text{def}}{=} \frac{1}{2\pi i} \int\limits_{C_+} f(z) dz$$

gde je C kontura oko z_0 koja pripada oblasti u kojoj je funkcija $f(z)$ analitička i koja u svojoj unutrašnjosti nema drugih singulariteta osim z_0.



Računanje reziduuma:

1° z_0 je **pol prvog reda:** $\lim\limits_{z\to z_0}(z-z_0)f(z)\neq 0.$

Tada je $f_1(z)=\begin{cases}(z-z_0)f(z), & z\neq 0\\ \lim\limits_{z\to z_0}(z-z_0)f(z), & z=z_0\end{cases}$ analitička

pa je, prema prvoj Košijevoj formuli,

$$\underbrace{f_1(z_0)}_{\lim\limits_{z\to z_0}(z-z_0)f(z)}=\frac{1}{2\pi i}\int\limits_{C_+}\frac{f_1(z)}{z-z_0}dz=\underbrace{\frac{1}{2\pi i}\int\limits_{C_+}\frac{(z-z_0)f(z)}{z-z_0}}_{Res[f(z),z_0]}$$

tj.

$$\boxed{Res[f(z), z_0] = \lim_{z\longrightarrow z_0}(z - z_0)f(z)}$$

Primer: $f(z) = \frac{z^2}{(z-1)(z+1)}$

$z = 1$, $z = -1$ polovi prvog reda
$Res[f(z), 1] = \lim_{z\longrightarrow 1}(z - 1)\frac{z^2}{(z-1)(z+1)} = \frac{1}{2}$
$Res[f(z), -1] = \lim_{z\longrightarrow -1}(z + 1)\frac{z^2}{(z-1)(z+1)} = -\frac{1}{2}$

2° z_0 je **pol** $n-$**tog reda:** $\lim_{z\longrightarrow z_0}(z - z_0)^n f(z) \neq 0$

Tada je $f_1(z) = \begin{cases} (z - z_0)^n f(z), & z \neq 0 \\ \lim\limits_{z\longrightarrow z_0}(z - z_0)^n f(z), & z = z_0 \end{cases}$ analitička

pa je, prema drugoj Košijevoj formuli,

$$\underbrace{f_1^{(n-1)}(z_0)}_{\lim\limits_{z\longrightarrow z_0}((z-z_0)^n f(z))^{(n-1)}} = \frac{(n-1)!}{2\pi i}\int\limits_{C_+}\frac{f_1(z)}{z - z_0}dz =$$

$$= \underbrace{\frac{(n-1)!}{2\pi i}\int\limits_{C_+}\frac{(z - z_0)^n f(z)}{(z - z_0)^n}}_{(n-1)!Res[f(z),z_0]}$$

tj.

$$Res[f(z), z_0] = \frac{1}{(n-1)!} \lim_{z \to z_0} \left(((z - z_0)^n f(z))^{(n-1)} \right)$$

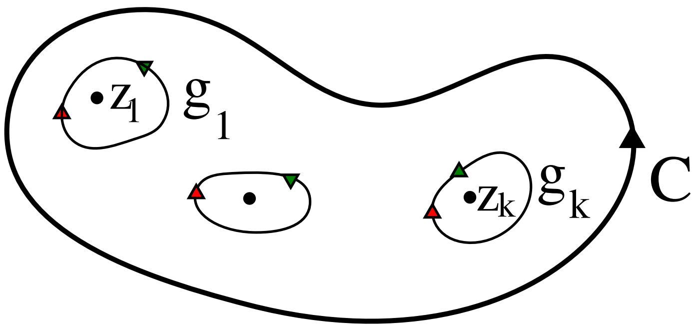
Primer: $f(z) = \frac{z^2}{(z+1)^2}$ $\quad z = -1$ je pol drugog reda

$$Res[f(z), -1] = \frac{1}{1!} \lim_{z \to -1} \left(((z + 1)^2 \frac{z^2}{(z+1)^2})' \right) =$$
$$= \lim_{z \to -1} 2z = -2$$

Teorema: Neka je jednostruko povezana oblast D ograničena konturom C i neka je $f(z)$ analitička na D i C, osim u singularnim tačkama $z_1, \ldots, z_k \in D$. Tada je

$$\int_{C^+} f(z)dz = 2\pi i \sum_{i=1}^{k} Res[f(z), z_i]$$

Dokaz:



Neka su γ_i krugovi oko z_i takvi da $\gamma_i \subset D$. Tada je $f(z)$ analitička na višestruko povezanoj oblasti D' koja je spolja ograničena konturom C, a iznutra konturama γ_i, $i = 1, \ldots, k$. Prema posledici Košijeve teoreme za višestruko povezane oblasti je

$$\int_{C^+} f(z)dz = \int_{\gamma_1} f(z)dz + \cdots + \int_{\gamma_k} f(z)dz =$$

$$= 2\pi i \left[\frac{1}{2\pi i} \int_{\gamma_1} f(z)dz + \cdots + \frac{1}{2\pi i} \int_{\gamma_k} f(z)dz \right] =$$

$$= 2\pi i \sum_{i=1}^{k} Res[f(z), z_i]. \qquad \longrightarrow$$

Domaći: Izračunati

$$\int\limits_{C^+} \frac{e^z}{(z^2+\pi^2)^2}\,dz, \quad \text{gde je } C : |z| = 4$$

LAPLASOVA TRANSFORMACIJA

1. Osnovni pojmovi

$f(t) \stackrel{\mathcal{L}}{\to} F(s),\ t \in R,\ s \in C$:

$$\boxed{\mathcal{L}(f(t)) = F(s) \stackrel{\text{def}}{=} \int\limits_0^\infty e^{-st} f(t)dt}$$

Napomena: $\int\limits_0^\infty e^{-st} f(t)dt = \lim\limits_{T\to\infty} \int\limits_0^T e^{-st} f(t)dt$

Oblast definisanosti funkcije $F(s)$: skup vrednosti $s \in C$ za koje $\int\limits_0^\infty e^{-st} f(t)dt$ konvergira.

Primer: $\mathcal{L}(e^{t^2})$ ne postoji jer $\int\limits_0^\infty e^{-st} e^{t^2} dt$ divergira $\forall s \in C$.

Klasa $E(a)$: $f(t) \in E(a)$ ako važi:

1° $f(t)$ je definisana na $[0, \infty)$

2° $f(t)$ ima najviše konačno mnogo prekida prve vrste na svakom konačnom podintervalu intervala $[0, \infty)$

3° $f(t)$ je eksponencijalnog reda rasta:
$\exists M > 0, a \in R \;\; t.d. \;\; |f(t)| \leq Me^{at}, \forall t \in [0, \infty)$

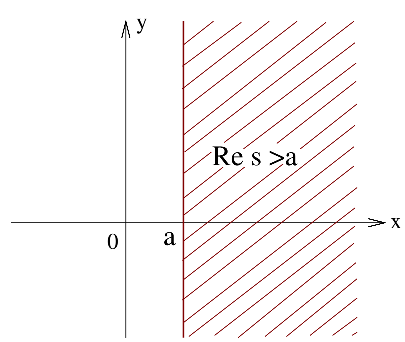
Primer: $f(t) = \sin t$

1° definisana je na $[0, \infty)$

2° neprekidna je na $[0, \infty)$

3° $|\sin t| \leq 1 \quad (M = 1, a = 0)$

Teorema (dovoljni uslovi za egzistenciju $\mathcal{L}(f(t))$): Neka $f(t) \in E(a)$. Tada je $F(s) = \mathcal{L}(f(t))$ definisana za $Re\, s > a$.



Dokaz: Teoreme o nesvojstvenim integralima:

T1: Ako je $0 \leq f_1(x) \leq f_2(x),\ x \geq a$, tada

$$\int_a^\infty f_2(x)dx \text{ konvergira } \Rightarrow \int_a^\infty f_1(x)dx \text{ konvergira}$$

T2: $\displaystyle\int_a^\infty |f_1(x)|dx$ konvergira $\Rightarrow \displaystyle\int_a^\infty f_1(x)dx$ konvergira

Prema T2 dovoljno je ispitati konvergenciju integrala $\int_0^\infty |e^{-st}f(t)|dt$. Kako je

$$0 \leq |\underbrace{e^{-st}f(t)}_{f_1}| \leq |e^{-st}|Me^{at} = |e^{-(\alpha+i\beta)t}|Me^{at} =$$

$$= e^{-\alpha t}Me^{at} = \underbrace{Me^{(a-\alpha)t}}_{f_2},$$

Prema T1 dovoljno je ispitati konvergenciju

$$\int_0^\infty Me^{(a-\alpha)t}dt = \frac{M}{a-\alpha}e^{(a-\alpha)t}|_0^\infty =$$

$$= \frac{M}{a-\alpha}[\underbrace{\lim_{T\to\infty} e^{(a-\alpha)t}}_{=0 \text{ za } a-\alpha<0} - 1] = -\frac{M}{a-\alpha},\ a - \alpha < 0, \text{ tj. za}$$

$Re\, s > a$.

Za $Re\, s > 0$ konvergira $\int\limits_0^\infty Me^{(a-\alpha)t}dt \;\Rightarrow\;$ konvergira $\int\limits_0^\infty |e^{-st}f(t)|dt \;\Rightarrow\;$ konvergira $\int\limits_0^\infty e^{-st}f(t)dt$, tj. $F(s)$ je definisana za $Re\, s > a$.

2. Laplasova transformacija nekih funkcija

1° $\mathcal{L}(\cos t) = \int\limits_0^\infty e^{-st}\cos t dt = \int\limits_0^\infty e^{-st}\,\frac{e^{it}+e^{-it}}{2}dt =$

$$= \frac{1}{2}\left[\int\limits_0^\infty e^{(-s+i)t}dt + \int\limits_0^\infty e^{(-s-i)t}dt\right] =$$

$$= \frac{1}{2}\left[\frac{1}{-s+i}e^{(-s+i)t}\Big|_0^\infty + \frac{1}{-s-i}e^{(-s-i)t}\Big|_0^\infty\right] =$$

$$= \frac{1}{2}\Big[\frac{1}{-s+i}(\lim_{T\to\infty} e^{(-s+i)T} - 1) +$$
$$+ \frac{1}{-s-i}(\lim_{T\to\infty} e^{(-s-i)T} - 1)\Big]$$

$$s=\alpha+i\beta:\ e^{(-s+i)T}=e^{(-\alpha-i\beta+i)T}=e^{-\alpha T}e^{i(1-\beta)T}=$$

$$=\underbrace{e^{-\alpha T}}_{\to 0,\alpha>0}[\underbrace{\cos(1-\beta)T}_{\text{ogran.}}+i\underbrace{\sin(1-\beta)T}_{\text{ogran.}}]\to 0,$$

$$T\to\infty,\ \alpha>0$$

Slično: $e^{(-s-i)T}\to 0,\ T\to\infty,\ \alpha>0$

$$\mathcal{L}(\cos t)=\frac{1}{2}\left[\frac{1}{s-i}+\frac{1}{s+i}\right]=\frac{s}{s^2+1},\ \alpha=Re\,s>0$$

Domaći: $\mathcal{L}(\sin t)=\frac{1}{s^2+1},\ \alpha=Re\,s>0$. Dokazati.

$2^\circ\ \mathcal{L}(\mathbf{t^n})=\int\limits_0^\infty e^{-st}t^n\,dt=I_n$

$n=0:$

$I_0=\mathcal{L}(1)=\int\limits_0^\infty e^{-st}dt=-\frac{1}{s}e^{-st}\Big|_0^\infty=-\frac{1}{s}(\lim\limits_{T\to\infty}e^{-sT}-1)$

$s=\alpha+i\beta:$

$$e^{-sT}=e^{-\alpha T}(\cos\beta T-i\sin\beta T)\to 0,$$

$$T\to\infty,\ \alpha=Re\,s>0$$

$$I_0 = \frac{1}{s}, \quad Re\, s > 0$$

$n \geq 1$:

$$I_n = \int_0^\infty e^{-st} t^n \, dt = \begin{pmatrix} u = t^n & dv = e^{-st}dt \\ du = nt^{n-1}dt & v = -\frac{1}{s}e^{-st} \end{pmatrix} =$$

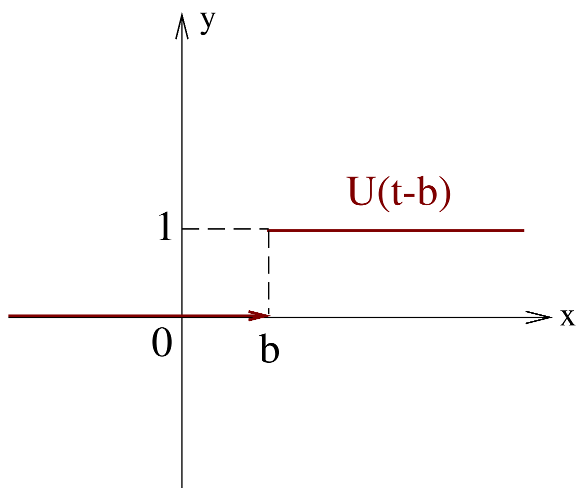
$$= -\left. \frac{t^n e^{-st}}{s} \right|_0^\infty + \frac{n}{s} \int_0^\infty e^{-st} t^{n-1} \, dt = \frac{n}{s} I_{n-1}, \; Re\, s > 0$$

$$\left(\lim_{T \to \infty} T^n e^{-sT} = 0, \; Re\, s > 0 \right)$$

Sledi:

$$I_n = \frac{n}{s} \cdot \frac{n-1}{s} \cdots \frac{1}{s} I_0 = \frac{n!}{s^n} \cdot \frac{1}{s} = \frac{n!}{s^{n+1}}, \; Re\, s > 0$$

3° $U(t-b) = \begin{cases} 0, & t < b \\ 1, & t \geq b \end{cases}$ jedinična odskočna funkcija



$$\mathcal{L}(U(t-b)) = \int\limits_0^{\infty} e^{-st} U(t-b) dt = \int\limits_0^{b} e^{-st} \cdot 0 \, dt +$$

$$+ \int\limits_b^{\infty} e^{-st} \cdot 1 \, dt = -\frac{1}{s} e^{-st} \Big|_b^{\infty} = -\frac{1}{s} [0 - e^{-sb}] = \frac{e^{-sb}}{s},$$

$$Re\, s > 0.$$

Domaći: Dokazati da je $\mathcal{L}(e^{at}) = \frac{1}{s-a}$

3. Osobine Laplasove transformacije

1° $f_1(t) \in E(a_1),\ f_2(t) \in E(a_2) \ \Rightarrow$

$$\boxed{\mathcal{L}(c_1 f_1 + c_2 f_2) = c_1 \mathcal{L}(f_1) + c_2 \mathcal{L}(f_2)} \quad \text{linearnost}$$

Dokaz:

$$\int_0^\infty e^{-st}(c_1 f_1 + c_2 f_2)dt = c_1 \int_0^\infty e^{-st} f_1 dt + c_2 \int_0^\infty e^{-st} f_2 dt,$$

$$Re\, s > \max\{a_1, a_2\}.$$

Primer:

$$\mathcal{L}(4t^3 - t + 1) = 4\mathcal{L}(t^3) - \mathcal{L}(t) + \mathcal{L}(1) = 4 \cdot \frac{3!}{s^4} - \frac{1}{s^2} + \frac{1}{s}.$$

2° $f(t) \in E(a),\ \mathcal{L}(f(t)) = F(s),\ Re\, s > a,\ b > 0 \Rightarrow$

$$\boxed{\mathcal{L}(f(bt)) = \tfrac{1}{b} F(\tfrac{s}{b})},\ Re\, s > ab$$

Dokaz: $\mathcal{L}(f(bt)) = \int_0^\infty e^{-st} f(bt)\, dt =$

$$(t' = bt \Rightarrow dt' = bdt)$$

$$= \frac{1}{b}\int_0^\infty e^{-\frac{s}{b}t'} f(t')dt' = \frac{1}{b}F(\frac{s}{b}), \quad Re\,(\frac{s}{b}) = \frac{Re\,s}{b} > a.$$

Primer: $\mathcal{L}(\sin t) = \frac{1}{s^2+1} = F(s)$

$$b > 0:\ \mathcal{L}(\sin bt) = \frac{1}{b}\left[\frac{1}{(\frac{s}{b})^2+1}\right] = \frac{b}{s^2+b^2}$$

$$b < 0:\ \mathcal{L}(\sin bt) = -\mathcal{L}(\sin(-bt)) = -\frac{-b}{s^2+b^2} = \frac{b}{s^2+b^2}$$

3° $f(t) \in E(a)$, $\mathcal{L}(f(t)) = F(s)$, $Re\,s > a \Rightarrow$

$$\boxed{\mathcal{L}(e^{bt}f(t)) = F(s-b)},\ Re\,s > a+b$$

Dokaz:

$$\mathcal{L}(e^{bt}f(t)) = \int_0^\infty e^{-st}e^{bt}f(t)dt = \int_0^\infty e^{-(s-b)t}f(t)dt =$$

$$= F(s-b), \quad Re\,(s-b) = Re\,s - b > a$$

Primeri:

$$\mathcal{L}(e^{bt}) = \mathcal{L}(e^{bt}\cdot 1) = \frac{1}{s-b}, \quad Re\,s > b\ (F(s) = \frac{1}{s})$$

$$\mathcal{L}(e^{bt}\sin at) = \frac{a}{(s-b)^2+a^2}, \quad Re\, s > b \ (F(s) = \frac{a}{s^2+a^2})$$

$$\mathcal{L}(e^{bt}\cos at) = \frac{s-b}{(s-b)^2+a^2}, \quad Re\, s > b \ (F(s) = \frac{s}{s^2+a^2})$$

4° $f(t) \in E(a)$, $\mathcal{L}(f(t)) = F(s)$, $Re\, s > a \Rightarrow$

$$\boxed{\mathcal{L}(f(t-b)U(t-b)) = e^{-bs}F(s)}, \quad Re\, s > a$$

Dokaz: $\mathcal{L}(f(t-b)U(t-b)) = \int\limits_0^b e^{-st} f(t-b)\cdot 0\, dt +$

$$+ \int\limits_0^b e^{-st} f(t-b)\cdot 1\, dt = \int\limits_0^\infty e^{-s(t'+b)} f(t')\cdot dt' = e^{-bs}F(s).$$

$$(t' = t-b \Rightarrow dt' = dt)$$

Primer: $\mathcal{L}(\sin 2t) = \frac{2}{s^2+4}$

$$\mathcal{L}(\sin 2(t-1)U(t-1)) = \frac{2e^{-s}}{s^2+4} \quad (b=1)$$

5° $f(t), f'(t) \in E(a)$, $\mathcal{L}(f(t)) = F(s)$, $Re\, s > a \Rightarrow$

$$\boxed{\mathcal{L}(f'(t)) = sF(s) - f(0)}, \quad Re\, s > a$$

Dokaz:

$$\mathcal{L}(f'(t))=\int\limits_0^\infty e^{-st}f'(t)dt=\begin{pmatrix} u=e^{-st} & dv=f'(t)dt \\ du=-se^{-st}dt & v=f(t) \end{pmatrix}=$$

$$=e^{-st}f(t)\Big|_0^\infty+s\int\limits_0^\infty e^{-st}f(t)dt=F(s)-f(0),\ Re\,s>a$$

$$(\lim_{T\to\infty} e^{-sT}f(T)=0,\ Re\,s>a).$$

Dokazuje se (indukcijom):

$$\mathcal{L}(f^{(n)}(t))=s^n-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$$

Primer:

$$y''-2y'+y=\sin t$$

$\mathcal{L}(y(t))=Y(s)\Rightarrow$

$$s^2Y(s)-sY(0)-y'(0)-2(sY(s)-y(0))+Y(s)=\frac{1}{s^2+1}$$

Diferencijalna jednačina prevedena je u algebarsku!

6° $f(t)\in E(a)$, $\mathcal{L}(f(t))=F(s)$, $Re\,s>a\Rightarrow$

$$\boxed{\mathcal{L}(\int\limits_0^t f(x)dx)=\frac{F(s)}{s}},\quad Re\,s>a$$

Dokaz: $\mathcal{L}(\int\limits_0^t f(x)dx) = \int\limits_0^\infty e^{-st}(\int\limits_0^t f(x)dx)dt =$

$$= \begin{pmatrix} u = \int\limits_0^t f(x)dx & dv = e^{-st}dt \\ du = f(t)dt & v = -\frac{1}{s}e^{-st} \end{pmatrix} = -\frac{1}{s}e^{-st}\int\limits_0^t f(x)dx\Big|_0^\infty +$$

$$+\frac{1}{s}\int\limits_0^\infty e^{-st}f(t)dt = \frac{F(s)}{s}, \quad Re\, s > a$$

$$(\lim_{T\to\infty} e^{-sT}\int_0^T f(x)dx = 0)$$

Dokazuje se (indukcijom):

$$\mathcal{L}(\underbrace{\int_0^t dt \cdots \int_0^t}_{n} f(t)dt) = \frac{F(s)}{s^n}, \quad Re\, s > a$$

7° $f_1(t)*f_2(t) \overset{\text{def}}{=} \int_0^t f_1(x)f_2(t-x)dx$ konvolucija

Komutativnost konvolucije: $f_1 * f_2 = f_2 * f_1$

$$f_2 * f_1 = \int_0^t f_2(x) f_1(t-x) dx = -\int_t^0 f_2(t-x') f_1(x') dx' =$$

$$(t - x = x' \Rightarrow dx = -dx') \qquad\qquad = f_1 * f_2$$

Slika konvolucije: $f_1 \in E(a_1),\ f_2 \in E(a_2) \Rightarrow$

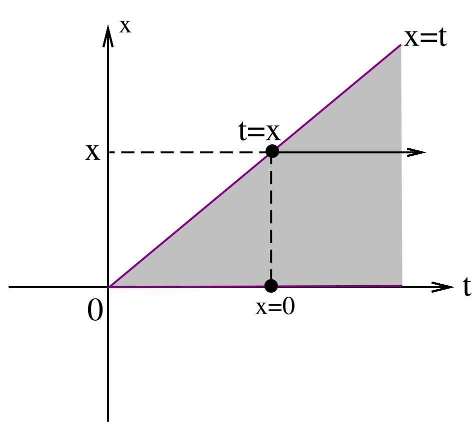
$$\boxed{\mathcal{L}(f_1 * f_2) = \mathcal{L}(f_1) \cdot \mathcal{L}(f_2)}, \quad Re\, s > \max\{a_1, a_2\}$$

Dokaz:

$$\mathcal{L}(f_1 * f_2) = \int_0^\infty e^{-st} \left(\int_0^t f_1(x) f_2(t-x) dx \right) dt =$$

$$= \int_0^\infty \int_0^t e^{-st} f_1(x) f_2(t-x) dx dt$$

Promenom poretka integracije (slika!)



dobija se

$$\mathcal{L}(f_1 * f_2) = \int_0^\infty dx \Big(\int_x^\infty e^{-st} f_1(x) f_2(t-x)\,dt\Big) =$$

$$= \int_0^\infty f_1(x)\,dx \Big(\int_x^\infty e^{-st} f_2(t-x)\,dt\Big)$$

Smenom $t - x = t'$, $dt = dt'$ u unutrašnjem integralu dalje se dobija

$$\mathcal{L}(f_1 * f_2) = \int_0^\infty f_1(x)\,dx \Big(\int_0^\infty e^{-s(x+t')} f_2(t')\,dt'\Big) =$$

$$= \int_0^\infty e^{-sx} f_1(x)\,dx \Big(\int_0^\infty e^{-st'} f_2(t')\,dt'\Big) = \mathcal{L}(f_1)\mathcal{L}(f_2).$$

8° $f(t) \in E(a)$, $\mathcal{L}(f(t)) = F(s)$, $Re\, s > a \Rightarrow$

$$\boxed{\mathcal{L}(tf(t)) = -F'(s)}, \quad Re\, s > a$$

Dokaz:

$$F'(s) = \frac{d}{ds} \int_0^\infty e^{-st} f(t)\,dt = -\int_0^\infty t e^{-st} f(t)\,dt = -\mathcal{L}(tf(t)).$$

Primer: $\mathcal{L}(t \sin t) = -\frac{d}{ds}\frac{1}{s^2+1} = \frac{2s}{(s^2+1)^2}$

Dokazuje se (indukcijom):

$$\mathcal{L}(t^n f(t)) = (-1)^n F^{(n)}(s)$$

9° $f(t), f(t)/t \in E(a)$, $\mathcal{L}(f(t)) = F(s)$, $Re\, s > a \Rightarrow$

$$\boxed{\mathcal{L}\left(\frac{f(t)}{t}\right) = \int\limits_s^\infty F(p)dp}, \quad Re\, s > a$$

Dokaz:

$$F(p) = \int\limits_0^\infty e^{-pt} f(t)dt$$

$$G(p) = \mathcal{L}\left(\frac{f(t)}{t}\right) = \int\limits_0^\infty e^{-pt}\frac{f(t)}{t}dt \Rightarrow$$

$$G'(p) = -\int\limits_0^\infty e^{-pt} f(t)dt = -F(p) \;\Big|\; \int\limits_S^s$$

$$\int\limits_S^s G'(p)dp = -\int\limits_S^s F(p)dp \Rightarrow G(s) - G(S) = \int\limits_s^S F(p)dp$$

$$\lim_{S\to\infty} G(S) = 0 \Rightarrow G(s) = \int\limits_s^\infty F(p)dp.$$

TABLICA LAPLASOVE TRANSFORMACIJE

a) **Tablica osobina:**

$f(t)$	$F(s)$
$\sum\limits_{i=1}^{n} c_i f_i(t)$	$\sum\limits_{i=1}^{n} c_i F_i(s)$
$f(bt)$	$\frac{1}{b}F(\frac{s}{b}),\ b > 0$
$e^{bt}f(t)$	$F(s-b)$
$f(t-b)U(t-b)$	$e^{-bs}F(s)$
$f'(t)$	$sF(s) - f(0)$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
$\int\limits_0^t f(x)dx$	$\frac{F(s)}{s}$
$\int\limits_0^t dt \cdots \int\limits_0^t f(t)dt$	$\frac{F(s)}{s^n}$
$f_1 * f_2$	$F_1(s)F_2(s)$
$tf(t)$	$-F'(s)$
$t^nf(t)$	$(-1)^nF^{(n)}(s)$
$\frac{f(t)}{t}$	$\int\limits_s^{\infty} F(p)dp$

b) **Laplasova transformacija nekih funkcija:**

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{bt}	$\frac{1}{s-b}$
$\sin bt$	$\frac{b}{s^2+b^2}$
$\cos bt$	$\frac{s}{s^2+b^2}$
$U(t-b)$	$\frac{e^{-bs}}{s}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
$\delta(t)$	1

Napomena: $\delta(t)$ je Dirakova funkcija:

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}$$

Osobina: $\int\limits_{-\infty}^{\infty} \delta(t)dt = 1$!!!

4. Inverzna Laplasova transformacija

$$\boxed{f(t) = \mathcal{L}^{-1}(F(s)) \Leftrightarrow F(s) = \mathcal{L}(f(t))}$$

$$(*) \begin{cases} f(t), g(t) \in E(a);\ f(t)=\mathcal{L}^{-1}(F(s)), g(t)=\mathcal{L}^{-1}(F(s)) \\ \text{Ne postoji interval } (c,d),\ c<d \text{ na kome je} \\ \qquad f(t) \neq g(t),\ t \in (c,d) \end{cases}$$

Primer: $f(t) = 1,\ g(t) = \begin{cases} 1, & t \neq 2 \\ 3, & t = 2 \end{cases}$

$$\left.\begin{array}{l} \mathcal{L}(f(t)) = \frac{1}{s} \\ \mathcal{L}(g(t)) = \frac{1}{s} \end{array}\right\} \Rightarrow \mathcal{L}^{-1}\left(\frac{1}{s}\right) \text{ nije jednoznačno definisana}$$

S obzirom na $(*)$, različite inverzne slike funkcije $F(s)$ ne mogu se "mnogo" razlikovati izmedju sebe. U praksi se $\mathcal{L}^{-1}(F(s))$ odredjuje iz tabele.

Primeri:

$$\mathcal{L}^{-1}\left[\frac{n!}{s^{n+1}}\right] = t^n$$

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] = \cos t$$

$$\mathcal{L}^{-1}\left[\frac{e^{-s}}{s^2+1}\right] = U(t-1)\sin(t-1)$$

1° $F(s) = \frac{P(s)}{Q(s)}$, $\;st\,P < st\,Q$;
$Q(s) = (s-s_1)\cdots(s-s_n)$ pol. sa realnim koeficijentima

$$\frac{P(s)}{Q(s)} = \underbrace{\frac{A_1}{s-a} + \cdots + \frac{A_k}{(s-a)^k}}_{s=a \text{ koren reda } k} + \cdots +$$

$$+\underbrace{\frac{M_1 s + N_1}{(s-\alpha)^2+\beta^2} + \cdots + \frac{M_r s + N_r}{((s-\alpha)^2+\beta^2)^r}}_{s=\alpha\pm i\beta \text{ koren reda } r}$$

$a)$ $\mathcal{L}^{-1}\left[\frac{A_1}{s-a}\right] = A_1\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = A_1 e^{at}$

$b)$ $\mathcal{L}^{-1}\left[\frac{A_k}{(s-a)^k}\right] = \frac{A_k}{(k-1)!}\mathcal{L}^{-1}\left[\frac{(k-1)1}{(s-a)^k}\right] = \frac{A_k}{(k-1)!}e^{at}t^{k-1}$

$c)\ \mathcal{L}^{-1}\left[\frac{M_1 s+N_1}{(s-\alpha)^2+\beta^2}\right]=\mathcal{L}^{-1}\left[\frac{M_1(s-\alpha)+M_1\alpha+N_1}{(s-\alpha)^2+\beta^2}\right]=$

$$=M_1\mathcal{L}^{-1}\left[\frac{s-\alpha}{(s-\alpha)^2+\beta^2}\right]+\frac{M_1\alpha+N_1}{\beta}\mathcal{L}^{-1}\left[\frac{\beta}{(s-\alpha)^2+\beta^2}\right]=$$

$$=M_1 e^{\alpha t}\cos\beta t+\frac{M_1\alpha+N_1}{\beta}e^{\alpha t}\sin\beta t$$

Primer: $F(s)=\frac{1}{s(s^2+1)}=\frac{A}{s}+\frac{Bs+C}{s^2+1}$

$$As^2+A+Bs^2+Cs=1\ \Rightarrow\ \begin{array}{l} A+B=0\\ C=0\\ A=1\end{array}\quad\Rightarrow B=-1$$

$$F(s)=\frac{1}{s}-\frac{s}{s^2+1}\ \Rightarrow\ f(t)=\mathcal{L}^{-1}(F(s))=1-\cos t$$

$d)\ \mathcal{L}^{-1}\left[\frac{M_r s+N_r}{((s-\alpha)^2+\beta^2)^r}\right]$ odredjuje se primenom osobina:

izvod slike: $\mathcal{L}^{-1}(F'(s))=-tf(t)$

konvolucija: $\mathcal{L}^{-1}(F_1(s)F_2(s))=f_1(t)*f_2(t)$

ili pomoću Melinove formule.

Primer: $G(s)=\frac{s}{(s^2+1)^2}=F'(s)$

$$F(s)=\int G(s)ds=\frac{1}{2}\int(s^2+1)^{-2}d(s^2+1)=-\frac{1}{2}\cdot\frac{1}{s^2+1}$$

$$\mathcal{L}^{-1}(F(s)) = -\frac{1}{2}\sin t \Rightarrow \mathcal{L}^{-1}(F'(s)) = -t(-\frac{1}{2}\sin t) =$$
$$= \frac{1}{2}t\sin t$$

Primer: $\mathcal{L}^{-1}\left(\frac{s}{(s^2+1)^2}\right) =$

$$= \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\cdot\frac{s}{s^2+1}\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) * \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) =$$

$$= \sin t * \cos t = \int_0^t \sin x \cos(t-x)dx = \frac{1}{2}\int_0^t [\sin(x+t-x)+$$

$$+\sin(x-t+x)]dx = \frac{1}{2}[\int_0^t \sin t dx + \int_0^t \sin(2x-t)dx] =$$

$$= \frac{1}{2}[\sin t \cdot x\Big|_0^t - \frac{1}{2}\cos(2x-t)\Big|_0^t] = \frac{1}{2}t\sin t$$

Domaći: Odrediti $\mathcal{L}^{-1}\left(\frac{1}{(s^2+1)^2}\right)$

4. Melinova formula

Teorema: Neka je:

1° $F(s)$ analitička u oblasti $Re\, s = x > a$

2° $\lim_{|s|\to\infty} F(s) = 0$ uniformno po $arg\, s$

3° $\forall x > a : \int\limits_{x-i\infty}^{x+i\infty} |F(s)| dy$ konvergira

Tada je za $Re\, s > a$ funkcija $F(s)$ Laplasova transformacija funkcije $f(t)$ i za $t > 0$ je

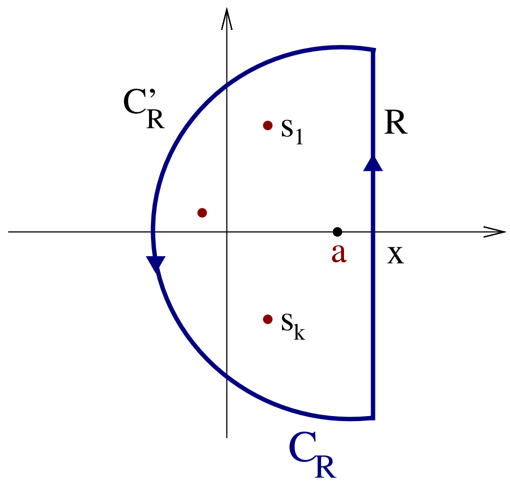
$$f(t) = \frac{1}{2\pi i} \int\limits_{x-i\infty}^{x+i\infty} e^{st} F(s) ds$$

(vrednost integrala ne zavisi od x, $x > a$!)

Teorema: Neka su ispunjeni uslovi $1^\circ, 2^\circ, 3^\circ$. Neka je, osim toga, $F(s)$ analitička za $Re\, s \leq a$ osim u tačkama $s_1, \ldots, s_k$. Tada je

$$f(t) = \sum_{i=1}^{k} Res[e^{st} F(s), s_i]$$

Dokaz:



$$\int\limits_{C_R^+} e^{st}F(s)ds = \underbrace{\int\limits_{C_R'}}_{\text{polukrug}} + \underbrace{\int\limits_{x-iR}^{x+iR}}_{\text{prečnik}}$$

S druge strane,

$$\int\limits_{C_R^+} e^{st}F(s)ds = 2\pi i \sum_{i=1}^{k} Res[e^{st}F(s), s_i] \Rightarrow$$

$$\int\limits_{C_R'} e^{st}F(s)ds + \int\limits_{x-iR}^{x+iR} e^{st}F(s)ds = 2\pi i \sum_{i=1}^{k} Res[e^{st}F(s), s_i]$$

Prelaskom na $\lim\limits_{R\to\infty}$ dobija se

$$\underbrace{\frac{1}{2\pi i}\int\limits_{x-i\infty}^{x+i\infty} e^{st}F(s)ds}_{f(t)} = \sum_{i=1}^{k} Res[e^{st}F(s), s_i]$$

jer $\int\limits_{C'_R} \to 0,\ R\to\infty$.

Primer: $F(s)=\frac{s}{(s^2+1)^2}$, $\;s=\pm i$ polovi reda 2

$$f(t) = Res[e^{st}F(s), i] + Res[e^{st}F(s), -i]$$

$$Res[e^{st}F(s), i] = \frac{1}{1!}\lim_{s\to i}\left((s-i)^2\frac{se^{st}}{(s-i)^2(s+i)^2}\right)' =$$

$$= \lim_{s\to i}\frac{(e^{st}+ste^{st})(s+i)^2 - 2(s+i)se^{st}}{(s+i)^4} =$$

$$= \frac{(e^{it}+ite^{it})2i - 2ie^{it}}{(2i)^3} = \frac{-2te^{it}}{-8i} = \frac{1}{4}t\sin t - \frac{1}{4}it\cos t$$

$$Res[e^{st}F(s), -i] = \frac{1}{1!}\lim_{s\to -i}\left((s+i)^2\frac{se^{st}}{(s-i)^2(s+i)^2}\right)' =$$

$$= \lim_{s \to -i} \frac{(e^{st} + ste^{st})(s-i)^2 - 2(s-i)se^{st}}{(s-i)^4} =$$

$$= \frac{(e^{-it} - ite^{-it})(-2i) + 2ie^{-it}}{(-2i)^3} = \frac{-2te^{-it}}{8i} =$$

$$= \frac{1}{4} it \cos t + \frac{1}{4} t \sin t$$

$$f(t) = \frac{1}{2} t \sin t$$

Primer: $F(s) = \frac{2}{s(s-1)(s^2+1)}$

$$f(t) = Res[e^{st}F(s), 0] + Res[e^{st}F(s), 1] +$$

$$+ Res[e^{st}F(s), i] + Res[e^{st}F(s), -i]$$

$$Res[e^{st}F(s), 0] = \lim_{s \to 0} s \frac{2e^{st}}{s(s-1)(s^2+1)} = \frac{2}{(-1) \cdot 1} = -2$$

$$Res[e^{st}F(s), 1] = \lim_{s \to 1} (s-1) \frac{2e^{st}}{s(s-1)(s^2+1)} = \frac{2e^t}{1 \cdot 2} = e^t$$

$$Res[e^{st}F(s), i] = \lim_{s\to i}(s-i)\frac{2e^{st}}{s(s-1)(s-i)(s+i)} =$$

$$= \frac{2e^{it}}{i(i-1)2i} = \frac{\cos t + i\sin t + i\cos t - \sin t}{2}$$

$$Res[e^{st}F(s), -i] = \frac{\cos t - i\sin t - i\cos t - \sin t}{2}$$

$$f(t) = -2te^{t} + \cos t - \sin t$$

5. Primene Laplasove transformacije

1° Linearna diferencijalna jednačina sa konstantnim koeficijentima sa početnim uslovom u 0:

$$a_0 x^{(n)} + a_1 x^{(n-1)} + \cdots + a_{n-1}x' + a_n x = f(t)$$

$$x(0) = x_0, x'(0) = x'_0, \ldots, x^{(n-1)}(0) = x_0^{(n-1)}$$

$$\mathcal{L}(x(t)) = X(s) \Rightarrow$$

$$a_0(s^n X(s) - s^{n-1}x_0 - \cdots - x_0^{(n-1)}) + \cdots + a_n X(s) = F(s)$$

$$X(s)\underbrace{(a_0s^n+\cdots+a_n)}_{Q(s)}=F(s)+P(s),\ st\,P\le n-1$$

$$X(s)=\frac{F(s)+P(s)}{Q(s)},\ x(t)=\mathcal{L}^{-1}(X(s))$$

Primer: $x''+x=1,\ x(0)=x'(0)=0$

$$s^2X(s)+X(s)=\frac{1}{s}\ \Rightarrow\ X(s)=\frac{1}{s(s^2+1)}$$

$$x(t)=Res[e^{st}X(s),0]+Res[e^{st}X(s),i]+Res[e^{st}X(s),-i]$$

$$Res[e^{st}X(s),0]=\lim_{s\to 0}s\frac{e^{st}}{s(s^2+1)}=1$$

$$Res[e^{st}X(s),i]=\lim_{s\to i}(s-i)\frac{e^{st}}{s(s-i)(s+i)}=\frac{e^{it}}{2i^2}=$$

$$=-\frac{1}{2}(\cos t+i\sin t)$$

$$Res[e^{st}X(s),-i]=\lim_{s\to i}(s+i)\frac{e^{st}}{s(s-i)(s+i)}=\frac{e^{-it}}{2i^2}=$$

$$=-\frac{1}{2}(\cos t-i\sin t)$$

$$x(t)=1-\cos t$$

2° Sistemi linearnih jednačina (prvog i višeg reda) sa konstantnim koeficijentima sa početnim uslovom u 0:

Primer:

$$\begin{array}{ll} x' - y = 0 & x(0) = 1 \\ y' + x = 0 & y(0) = 0 \end{array}$$

$$\left.\begin{array}{l} sX(s) - 1 - Y(s) = 0 \\ sY(s) + X(s) = 0 \end{array}\right\} \Rightarrow$$

$$\begin{aligned} X(s) &= -sY(s) \\ Y(s) &= -\frac{1}{s^2+1} \Rightarrow y(t) = -\sin t \\ X(s) &= \frac{s}{s^2+1} \Rightarrow x(t) = \cos t \end{aligned}$$

Formalni zapis:

$$\frac{dX}{dt} = AX + b(t), \quad X(0) = X_0$$

$$\mathsf{X}(s) = \mathcal{L}(X(t)) = \begin{bmatrix} \mathcal{L}(x_1(t)) \\ \vdots \\ \mathcal{L}(x_n(t)) \end{bmatrix}$$

$$\mathcal{L}(\frac{dX}{dt}) = s\mathsf{X}(s) - X_0 \Rightarrow s\mathsf{X}(s) - X_0 = A\mathsf{X}(s) + B(s) \Rightarrow$$

$$(sI - A)\mathsf{X} = B(s) + X_0$$

$$\mathsf{X}(s) = (sI - A)^{-1}(B(s) + X_0)$$

$$X(t) = \mathcal{L}^{-1}((sI - A)^{-1}(B(s) + X_0))$$

$$\boxed{X(t) = \int_0^t e^{A(t-x)} b(x)\, dx + e^{At} X_0}$$

Primer:

$$\begin{aligned} x'' - y'' + y' - x &= e^t - 2 \\ 2x'' - y'' - 2x' + y &= -t \end{aligned}$$

$$x(0) = y(0) = x'(0) = y'(0) = 0$$

$$\left.\begin{array}{l} s^2X(s) - s^2Y(s) + sY(s) - X(s) = \frac{1}{s-1} - \frac{2}{s} \\ 2s^2X(s) - s^2Y(s) - 2sX(s) + Y(s) = -\frac{1}{s^2} \end{array}\right\} \Rightarrow$$

$$X(s) = \frac{1}{s(s-1)^2},\ Y(s) = \frac{2s-1}{s^2(s-1)^2} \Rightarrow$$

$$x(t) = 1 - e^t + te^t, \quad y(t) = -t + te^t$$

3° Integro-diferencijalne jednačine:

Primer: $$y(t) + 4\underbrace{\int_0^t e^{x-t}(t-x)y(x)dx}_{te^{-t}*y(t)} = e^{-3t}$$

$\mathcal{L}(y(t)) = Y(s),\ \mathcal{L}(te^{-t}) = \frac{1}{(s+1)^2},\ \mathcal{L}(e^{-3t}) = \frac{1}{s+3}$

$$Y(s) + 4\cdot\frac{1}{(s+1)^2}Y(s) = \frac{1}{s+3} \Rightarrow$$

$$Y(s) = \frac{s^2+2s+1}{(s^2+2s+5)(s+3)}$$

Domaći: Odrediti $y(t)$